Investigation of the locking bandwidth in linear and circular arrays of mutually coupled oscillators, intended for microwave power combining

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Abstract: An array of N weakly coupled oscillators is investigated in two major configurations. The locking bandwidth for linear and circular arrays is assessed through an array coupling matrix, in a worst-case free-running frequency distribution of oscillators. The superior locking bandwidth for circular arrays is demonstrated.

1 Introduction

Power combining of synchronised oscillators is an efficient means of achieving higher output levels from solid-state sources especially for microwaves and millimetre-wave applications [1, 2]. Synchronisation or inter-injection locking of oscillators is obtained by establishment of coupling between these oscillators either by passive networks or by mutual coupling via an antenna array connected to the oscillators [3, 4]. If the difference between the free-running frequencies of the oscillators is greater than a threshold value called locking bandwidth, synchronisation would become impossible. This threshold is a function of the coupling coefficient (magnitude and phase), number of oscillators and also the arrangement of the oscillators in the array. Ring or circular arrangement of coupled oscillators has been studied recently [5, 6]. For injection locked oscillators or two mutually coupled oscillators, the locking bandwidth has been defined and calculated previously [7, 8]. In this paper we extend the definition of locking bandwidth and compute the locking bandwidth of N mutually coupled oscillators arranged in a linear or in a circular array configuration. We compare the locking bandwidths in a worst-case condition. It will be shown that the locking bandwidth of the circular array, according to our definition, is twice that of a linear one with the same number of oscillators and coupling coefficients. To confirm our theoretical results, a practical implementation of resistively coupled oscillator array is realised and their locking bandwidth is measured.

2 Mutually coupled oscillators array

Consider an array of oscillators in which each oscillator is coupled with its adjacent oscillator and not the other ones. This can be the case in active arrays in which the coupling between adjacent antennas is significant but the coupling between farther antennas is negligible. Two configurations are considered here. First, the linear array in which the adjacent oscillators are coupled but the first and the last elements are uncoupled (Fig. 1*a*). Secondly the circular array in which all the elements including the first and the last one are coupled (Fig. 1*b*). In other words all elements are coupled to two adjacent ones. The coupling can be either through electromagnetic fields of radiating antennas or through a passive circuit connecting adjacent oscillators. To study the problem, we consider a series resonant circuit model for each oscillator as in Fig. 2. $\boldsymbol{A}_n$ is the loop current phasor in oscillator n and $-R_D$ is the describing function for the active nonlinear resistance of the oscillator. For simplicity the reactance of the active element is absorbed into the reactance of the passive part of the circuit. This is justified as long as the reactance variations of the active element is small compared to the amplitude of the oscillations. Let us consider $V_{inj,n}$ as the injection voltage phasor in oscillator n as a vector summation of all other oscillators induction into this oscillator [9]:

$$V_{inj,n} = \sum_{\substack{m=1 \\ m\neq n}}^{N} k_{nm} \cdot V_m = \sum_{\substack{m=1 \\ m\neq n}}^{N} \varepsilon_{nm} R_L e^{-j\Phi_{nm}} A_m e^{j\theta_m} \qquad (1)$$

where $\varepsilon_{nm}e^{-j\Phi_{nm}}$ is the coupling coefficient between oscillators m and n, A_m is the amplitude and θ_m is the phase of the mth oscillator current. Here, each oscillator has a free-running frequency which is determined by C_n and L_n. The values of ω_n are slightly different for each oscillator but are distributed around an average value say ω_0. As normally the same active elements and the same loads are used in the oscillator arrays, we consider that the oscillation amplitudes are approximately the same and equal to A_0. Assuming that $V_{inj,n}$ is small, the amplitude and the phase of oscillations will change slightly about the free-running values:

$$V_n(t) = R_L A_n(t) \cdot e^{j(\omega_n t + \varphi(t))} = R_L A_n(t) \cdot e^{j\theta(t)} \qquad (2)$$

such that

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} \ll \omega_0, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} \ll \omega_0 \qquad (3)$$

with this assumption, writing KVL in each oscillator's circuit, we come up with the following differential

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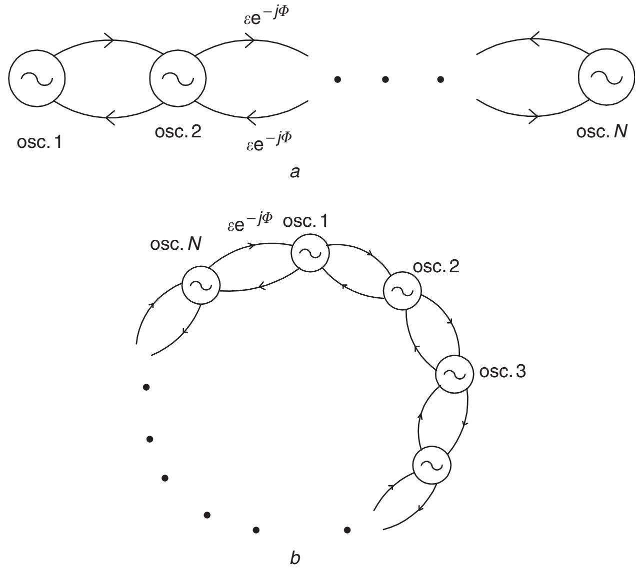

Fig. 1 *Linear and circular arrays of mutually coupled oscillators*
a Coupled adjacent oscillators with first and last elements uncoupled
b All elements including first and last are coupled

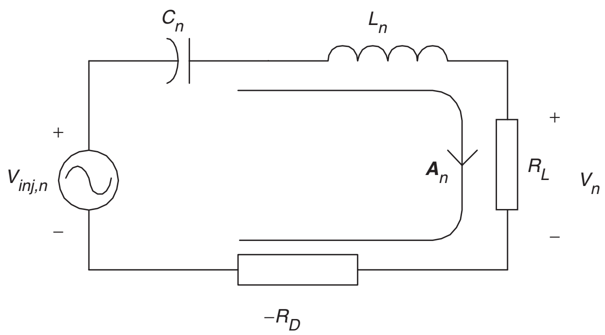

Fig. 2 *Series model for nth oscillator*

equations:

$$\frac{dV_n}{dt} = V_n\left[\frac{-\omega_n}{2Q}\left(1-\frac{R_D}{R_L}\right)+j\omega_n\right]+\frac{\omega_n}{2Q}V_{inj,n}, \qquad n = 1, 2, \ldots, N \tag{4}$$

Here, ω_n is the free-running frequency of the nth oscillator and $Q = L\omega_0/R_L$ is the external quality factor with ω_0 being the average value of ω_n. Using (2) and separating the real and imaginary parts of (4), we arrive at two sets of differential equations for amplitudes $A_n(t)$ and phases $\theta_n(t)$, respectively,

$$\frac{dA_n}{dt} = \frac{\omega_n}{2Q}A_n\left(1-\frac{R_D}{R_L}\right)+\frac{\omega_n}{2Q}A_n\cdot \mathrm{Re}\left\{\frac{V_{inj,n}}{V_n}\right\} \tag{5a}$$

$$\frac{d\theta_n}{dt} = \omega_n + \frac{\omega_n}{2Q}\mathrm{Im}\left\{\frac{V_{inj,n}}{V_n}\right\} \tag{5b}$$

$-R_D$ is a function of oscillation amplitude. Assuming weak coupling between oscillators, which is usually valid in many active antenna arrays, the oscillation amplitude would not differ significantly from the free-running value but the phase would change according to (5*b*). By substitution of (1) into (5*b*) obtain a set of equations which describe the phase and frequency change of each oscillator and the locking bandwidth of the array:

$$\frac{d\theta_n}{dt} = \omega_n + \frac{\omega_n}{2Q}\sum_{\substack{m=1\\ m\neq n}}^{N}\varepsilon_{nm}\frac{A_m}{A_n}\cdot\sin(\theta_m-\theta_n-\Phi_{nm}) \tag{6}$$

3 Locking bandwidth of the array

The oscillators, having different free-running frequencies, in an array, are locked once they all oscillate at the same frequency, with a constant phase difference between each other. The locking bandwidth is defined as the maximum free-running frequency difference between any pair of oscillators in the array, for which the array can be locked. Here, we assume N oscillators of the same construction in the array. In this case, A_m/A_n would be approximately equal to unity. As in the oscillator arrays, the main objective of the array construction is the in phase power combining (for maximum boresight radiation), here we impose the in-phase coupling between the oscillators, i.e. $\Phi_{mn} = 2k\pi$, throughout our analysis. In a uniform linear array $\varepsilon_{mn} = \varepsilon$ for $|m-n| = 1$ and $\varepsilon_{mn} = 0$ otherwise, see Fig. 1*a*. In a uniform circular array $\varepsilon_{mn} = \varepsilon$ for $|m-n| = 1$ and $\varepsilon_{mn} = 0$ for $|m-n| \neq 1$ except for $\varepsilon_{N1} = \varepsilon$, see Fig. 1*b*. We look for a steady-state solution of the locking problem. After locking, all $d\theta_n/dt$ would become equal to a common frequency of oscillation, by subtracting one of each two consecutive equations in (6) from the other we obtain a set of $N-1$ equations, for a linear array, and a set of N equations, for a circular array, as follows [9]:

$$\alpha[A]\cdot \boldsymbol{S} = -\boldsymbol{\Omega} \tag{7}$$

or

$$\boldsymbol{S} = \frac{-1}{\alpha}[A^{-1}]\cdot\boldsymbol{\Omega}$$

where $\boldsymbol{S}$ is the oscillator's phase shift vector $\boldsymbol{S} = [\sin\Delta\theta_1\ \sin\Delta\theta_2\ \ldots\ \sin\Delta\theta_{N-1}]^T$ in which $\Delta\theta_n = \theta_n - \theta_{n+1}$, $\boldsymbol{\Omega}$ is the frequency difference vector, $\boldsymbol{\Omega} = [\Omega_1\ \ \Omega_2\ \ldots\ \Omega_{N-1}]^T$, $\Omega_n = \omega_n - \omega_{n+1}$ and $\alpha = \varepsilon\omega_0/2Q$.

For a linear array, $[A]$ is the array coupling matrix of $(N-1)\times(N-1)$ dimension which has the following form:

$$[A] = \begin{bmatrix} -2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & -2 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & 0 \\ 0 & \cdot & \cdot & \cdot & 1 & -2 & 1 \\ 0 & 0 & \cdot & \cdot & 0 & 1 & -2 \end{bmatrix}_{(N-1)\times(N-1)} \tag{8}$$

For a circular array, $[A]$ is a $N\times N$ matrix which has the following form:

$$[A] = \begin{bmatrix} -2 & 1 & 0 & \cdot & \cdot & \cdot & 1 \\ 1 & -2 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & 0 \\ 0 & \cdot & \cdot & \cdot & 1 & -2 & 1 \\ 1 & 0 & \cdot & \cdot & 0 & 1 & -2 \end{bmatrix}_{N\times N} \tag{9}$$

Here, $\Omega_N = \omega_N - \omega_1$ and $\Delta\theta_N = \theta_N - \theta_1$ in circular array. Assuming ω_0 as the mean free-running frequency of the oscillators, we look for the worst-case distribution of the free-running frequencies for which there exists a solution for values of $\Delta\theta_i$ and we compute the locking bandwidth consequently. It is evident that equation (7) has a solution only when all members of the phase difference vector $\boldsymbol{S}$ are less than or equal to unity. As such, we look for a distribution of ω_i which, once $\boldsymbol{\Omega}$ is multiplied by $[A]^{-1}$, can result in the largest absolute value in one of the rows of $\boldsymbol{S}$. By some inspection and noticing that elements of the $[A]^{-1}$ for linear array are

$$[A^{-1}]_{ij} = \alpha_{ij}^{-1} = \frac{j(i-N)}{N} \quad i \geq j \qquad (10)$$
$$[A^{-1}]_{ji} = [A^{-1}]_{ij}$$

we come across a step-like distribution for ω_i as depicted in Fig. 3. For this distribution, the norm of $\boldsymbol{S}$ will be maximum. Such a distribution of free-running frequencies results in an $\boldsymbol{\Omega}$ as follows:

$$\begin{array}{rcl} \boldsymbol{\Omega} & = & [\; 0 \;\; 0 \; \cdots \; 0 \; 1 \; 0 \; \cdots \;\; 0 \;] \\ \text{column number} & \rightarrow & \;\;\; 1 \;\; 2 \; \cdots \;\; N/2 \;\; \cdots \; N-1 \end{array} \qquad (11)$$

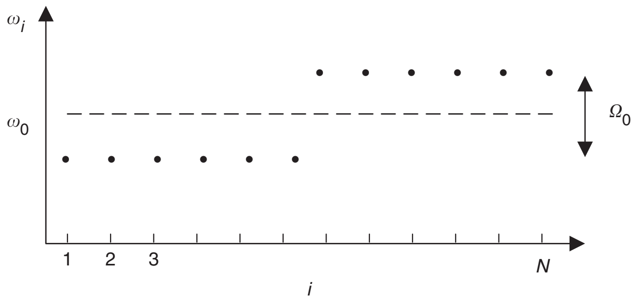

Fig. 3 *Arrangement of free-running frequencies for which $\|\bar{S}\|_\infty$ is maximum (worst case distribution)*

As $[A^{-1}]_{ij}$ is the largest member of $[A]^{-1}$ the $(N/2)$th row of $\boldsymbol{S}$ will have the value of $(-1/\alpha)\Omega_0(N/4)$ which is the component of this vector which has the largest magnitude. Every other distribution of free-running frequencies will result in a smaller value for every row of $\boldsymbol{S}$. As a result

$$\frac{1}{\alpha}\Omega_0\frac{N}{4} \leq 1 \quad \text{or} \quad \Omega_0 \leq \frac{4\alpha}{N} \qquad (12)$$

This frequency distribution is the worst case. Here, the worst case means that, in any other distribution, if (12) is satisfied locking would be certain. Time domain simulations confirm this result. In Fig. 4 the numerical solution of differential equations of (6) with random initial phases for oscillators is shown for an array of size $N = 6$ and the step-like frequency distribution is assumed as in Fig. 3 with Ω_0 just below the $4\alpha/N$ value for having a locked condition. The other parameters are $Q = 10$, $\varepsilon = 0.1$ and $f_0 = 1$ GHz. It is obvious that the phase difference reaches values which are consistent with (7). It is noticeable that, although a solution for $\boldsymbol{S}$ corresponds to 2^{N-1} different solutions for $\boldsymbol{\Delta\theta}$, only one of them results in a stable solution.

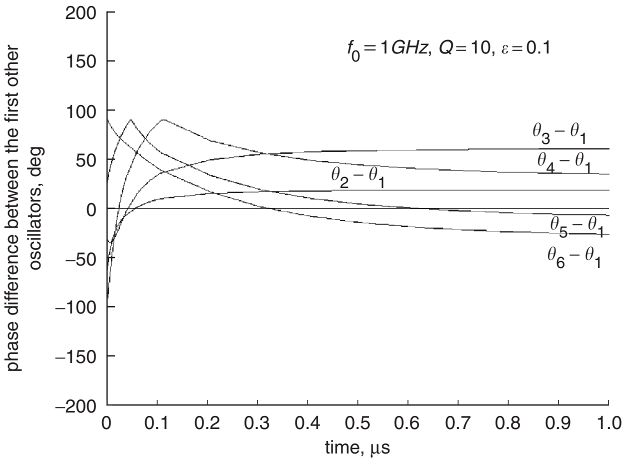

Fig. 4 *Numerical simulation of differential equations (6) for linear array of steplike frequency distribution with Ω_0 just below $4\alpha/N = 3.33$ MHz, showing locked (constant phase) condition*

4 Circular array locking bandwidth

Hereafter, we present a relatively long mathematical reasoning to arrive at the locking bandwidth of a circular array. For this array again we have

$$\alpha[A] \cdot \boldsymbol{S} = -\boldsymbol{\Omega} \qquad (13)$$

For this arrangement of oscillators, $[A]$ has the form of (9) with N rows and N columns. In addition $\boldsymbol{S}$ and $\boldsymbol{\Omega}$ have N rows in this case. However the determinant of $[A]$ is zero, consequently it cannot be inversed. For the existence of a solution of equation (13), it is sufficient for the vector $\boldsymbol{\Omega}$ to be perpendicular to the null space of $[A^*]$ (which is the transposed conjugate of $[A]$), see reference [10] for example. The null space of $[A^*]$ (which is equal to that of $[A]$) includes all vectors in the direction of $\boldsymbol{S}_{||} = [1 \;\; 1 \;\; \cdots \;\; 1]^T$, and we have $[A]\boldsymbol{S}_{||} = \boldsymbol{0}$. As for any combination of free-running frequencies we have the property $\sum_{i=1}^{N} \Omega_i = 0$, we conclude that the vector $\boldsymbol{\Omega}$ is normal to $\boldsymbol{S}_{||}$. Although (13) has an infinite number of solutions, and a new solution can be obtained by adding a vector in the direction of $\boldsymbol{S}_{||}$ to any previous solution, among them only one is physically valid. The acceptable solution has the property that all the element magnitudes of $\boldsymbol{S}$ must be equal to or less than one, and on the other hand the property that $\sum_{i=1}^{N} \Delta\theta_i = 0$ must also be satisfied. Given the symmetry of the problem we can try to find a solution. For this purpose we assume a nonreciprocal coupling between the first and the last oscillators, different from the others, say ε' instead of ε. In this case the matrix $[A]$ becomes:

$$[A'] = \begin{bmatrix} -2 & 1 & 0 & \cdot & \cdot & \cdot & p \\ 1 & -2 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ \cdot & \cdot & & & & & 0 \\ 0 & \cdot & \cdot & \cdot & 1 & -2 & 1 \\ 1 & 0 & \cdot & \cdot & 0 & 1 & -2 \end{bmatrix}_{N \times N} \qquad (14)$$

At the limit as p, in the above matrix, approaches 1 the matrix $[A']$ takes the original form of $[A]$. Let us present the lower and upper triangular factorisation of $[A']$

$$[L'] = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -1/2 & 1 & 0 & & & & & 0 \\ 0 & -2/3 & 1 & \cdot & \cdot & & & \cdot \\ 0 & 0 & -3/4 & 1 & \cdot & & & \cdot \\ 0 & & & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & -(N-1)/N & 1 & 0 \\ -1/2 & -1/3 & -1/4 & \cdot & \cdot & \cdot & -1 & 1 \end{bmatrix}_{N \times N} \tag{15}$$

$$[U'] = \begin{bmatrix} -2/1 & 1 & 0 & \cdot & \cdot & \cdot & 0 & p \\ 0 & -3/2 & 1 & 0 & \cdot & \cdot & \cdot & p/2 \\ 0 & 0 & -4/3 & 1 & 0 & & \cdot & p/3 \\ \cdot & & 0 & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & 1 & 0 & \cdot \\ \cdot & & & & \cdot & -(N-1)/(N-2) & 1 & p/(N-2) \\ \cdot & & & & & 0 & -N/(N-1) & 1 + p/3 \\ 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & p - 1 \end{bmatrix} \tag{16}$$

where $[A'] = [L'] \times [U']$. The matrix $[L']$ is independent of p. The inverses of $[L']$ and $[U']$ are as follows:

$$[L']^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 1/2 & 1 & 0 & & & & & 0 \\ 1/3 & 2/3 & 1 & 0 & & & & 0 \\ 1/4 & 2/4 & 3/4 & 1 & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & 0 & 0 \\ 1/(N-1) & 2/(N-1) & \cdot & \cdot & \cdot & (N-2)/(N-1) & 1 & 0 \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 \end{bmatrix}_{N \times N} \tag{17}$$

$$[U']^{-1} = \begin{bmatrix} -1/2 & -1/3 & -1/4 & \cdot & \cdot & \cdot & -1/N & \dfrac{1 + (N-1)p}{N(p-1)} \\ 0 & -2/3 & -2/4 & \cdot & \cdot & \cdot & -2/N & \dfrac{2 + (N-2)p}{N(p-1)} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \dfrac{3 + (N-3)p}{N(p-1)} \\ 0 & 0 & & \cdot & \cdot & \cdot & \cdot & \\ \cdot & & & & \cdot & \cdot & \cdot & \\ \cdot & & & & & \cdot & \cdot & \\ \cdot & & & & & 0 & -(N-1)/N & \dfrac{(N-1) + p}{N(p-1)} \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & \dfrac{1}{(p-1)} \end{bmatrix}_{N \times N} \tag{18}$$

Now $\boldsymbol{S}$ is computed from

$$\boldsymbol{S} = \frac{-1}{\alpha} [U']^{-1} [L']^{-1} \boldsymbol{\Omega} \tag{19}$$

See the Nth row of $[L']^{-1}$ which is multiplied by $\boldsymbol{\Omega}$ and forms a vector whose Nth row is zero, because $\Sigma_{i=1}^{N} \Omega_i = 0$.

As the multiplication of $[U']^{-1}$ by the resulting vector produces the rows of $\boldsymbol{S}$, the contribution of the Nth column of $[U']^{-1}$ in those rows would become apparently zero and for the Nth row of $\boldsymbol{S}$, $\sin\Delta\theta_N$ becomes zero. However, at the limit, as p approaches unity, the elements of the last column of $[U']^{-1}$ tend to infinity. Consequently, multiplication of them by zeros can result in a limit value. In other words, by rearranging $[U']^{-1}[L']^{-1}$, we can split the resulting matrix into two parts. One part independent of p, namely $[B]$, the other part dependent on p, namely $[C]$:

$$[U']^{-1}[L']^{-1} = [B] + [C] \tag{20}$$

in which the elements of $[B]$ are

$$[B]_{ij} = -ij\left(\frac{1}{j(j+1)} + \frac{1}{(j+1)(j+2)} + \cdots + \frac{1}{(N-1)N}\right)$$

$$= -ij\left(\frac{N-1}{N} - \frac{j-1}{j}\right) \quad j \geq i \tag{21}$$

and

$$[B]_{ji} = [B]_{ij}$$

As an example for $N = 6$ the matrix $[B]$ would have the following form:

$$[B] = \frac{-1}{6}\begin{bmatrix} 5 & 4 & 3 & 2 & 1 & 0 \\ 4 & 8 & 6 & 4 & 2 & 0 \\ 3 & 6 & 9 & 6 & 3 & 0 \\ 2 & 4 & 6 & 8 & 4 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

The matrix $[C]$ can be expressed as the product of two matrices

$$[C] = [C'] \times [I] \tag{23}$$

where

$$[C'] = \begin{bmatrix} \frac{1+(N-1)p}{N(p-1)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{2+(N-2)p}{N(p-1)} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{N-1+p}{N(p-1)} & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{(p-1)} \end{bmatrix}$$

and

$$[I] = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \ddots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

The matrices $[B]$ and $[C]$ multiplied by $\boldsymbol{\Omega}$ separately result in the vector $\boldsymbol{S}$. At the limit as p approaches 1, all diagonal elements in $[C']$ approach the same value, i.e. $1/(1-p)$ and tend to infinity consequently. On the other hand the multiplication of matrix $[I]$ by $\boldsymbol{\Omega}$ results in a null vector. Now the multiplication of this null vector by the matrix $[C']$ results in a constant vector at the limit. We conclude that the role of matrix $[C]$ in the solution for $\boldsymbol{S}$ is the addition of an arbitrary constant to all elements of vector $\boldsymbol{S}$, in other words an arbitrary vector in direction $\boldsymbol{S}_{||}$ can be added to $\boldsymbol{S}$, to produce a new solution.

At this stage, we should consider a frequency distribution such that $([B] + [C]) \cdot \boldsymbol{\Omega}$ produces the largest values for rows of $\boldsymbol{S}$. In this configuration (the circular array), as in the linear array, the worst case for frequency distribution would be the one depicted in Fig. 3. However, in this case, the vector $\boldsymbol{\Omega}$ would have the form of

$$\boldsymbol{\Omega} = \Omega_0[0 \quad 1 \quad 0 \quad \cdots \quad 0 \quad -1 \quad 0]^T$$

In this vector, between values $+1$ and -1, there would be exactly $(N/2) - 1$ zeros. Multiplying $[B]$ by $\boldsymbol{\Omega}$ the rows of $\boldsymbol{S}$ would have the following forms:

$$\boldsymbol{S}_i = -\frac{\Omega_0}{\alpha}\left([B]_{i,\frac{N}{2}-1} - [B]_{i,N-1}\right)$$

$$= \begin{cases} \frac{\Omega_0}{\alpha} \cdot \frac{i}{2}, & \text{for} \quad i \leq \frac{N}{2} - 1 \\ \frac{\Omega_0}{\alpha}\left(\frac{i}{2} - \frac{N}{2} + 1\right), & \text{for} \quad \frac{N}{2} - 1 < i < N - 1 \\ \frac{-\Omega_0}{2\alpha}, & \text{for} \quad i = N - 1 \end{cases} \tag{24}$$

The largest element in these rows has a magnitude of $(\Omega_0/\alpha) \cdot (N-2)/4$. Adding a constant to all elements $\boldsymbol{S}_i$ will result in a solution for $\boldsymbol{S}$. According to equation (24), the row number $(N/2) - 1$ has the largest value, and the row number $N - 1$ has the smallest value $-\Omega_0/2\alpha$. Now we add a value C to all elements of $\boldsymbol{S}_i$ such that the smallest and largest elements of $\boldsymbol{S}$ lay between -1 and $+1$:

$$\frac{\Omega_0}{\alpha} \cdot \frac{N-2}{4} + C \leq 1$$

$$-1 \leq \frac{-\Omega_0}{2\alpha} + C \tag{25}$$

or

$$-1 + \frac{\Omega_0}{2\alpha} \leq C \leq 1 - \frac{\Omega_0}{\alpha} \cdot \frac{N-2}{4}$$

therefore

$$-1 + \frac{\Omega_0}{2\alpha} \leq 1 - \frac{\Omega_0}{\alpha} \cdot \frac{N-2}{4} \quad \text{or} \quad \Omega_0 \leq \frac{8\alpha}{N} \tag{26}$$

We conclude that the locking bandwidth for the worst-case free-running frequency distribution of oscillators in a circular array would be twice that of a linear array of the same oscillators with the same coupling. Numerical simulations confirm this result. Figures 4 and 5 show the results of numerical simulation of phase differential equations governing the linear and circular arrays. Constant

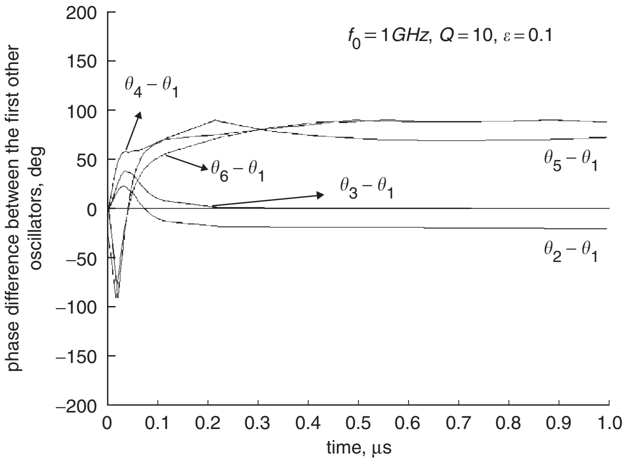

Fig. 5 *Numerical simulation of differential equations (6) for circular array of steplike frequency distribution with Ω_0 just below $8\alpha/N = 6.66$ MHz, showing locked (constant phase) condition*

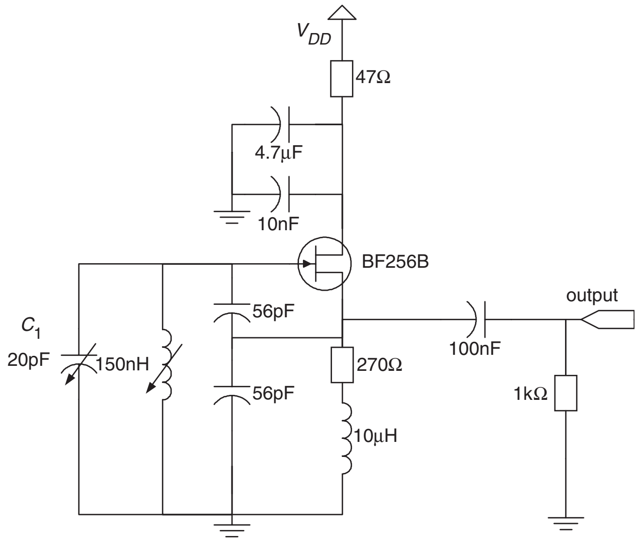

Fig. 6 *Unit cell Colpitts oscillator used in simulation and experiment*

phase difference with respect to time shows that the locking has materialised. In Fig. 5 the phase differences between oscillators of a circular array with the same parameters as in Fig. 4 are plotted, the synchronisation is possible while Ω_0 has been doubled.

5 Numerical simulation and experimental verification

To confirm the theoretical results we have devised an array of 4 to 6 oscillators using a BF256B field effect transistor in Colpitts configuration. The configuration of the unit cell of the array of oscillators is depicted in Fig. 6. The operation of the oscillators was simulated using Agilent's ADS software. These oscillators oscillate at VHF range in the vicinity of 58.2 MHz, giving an amplitude of 2.75 V on a 1 kΩ load. This frequency band allows us to adjust easily the oscillators' free-running frequencies by a variable capacitor, namely C_1. The oscillators free-running frequencies will be adjusted for the worst case according to the distribution in Fig. 3. For the sake of simplicity, the oscillators are coupled through series resistors R as shown in Fig. 7. The model for this type of coupling is depicted in Fig. 8.

The injected current for each oscillator and the coupling coefficient would be deduced as:

$$I_{inj1} = \frac{V_2}{R}, \; I_2 = \frac{V_2}{R_L} \quad \text{and} \quad \varepsilon_1 = \frac{I_{inj1}}{I_2} = \frac{R_L}{R}$$

$$I_{inj2} = \frac{V_1}{R}, \; I_1 = \frac{V_1}{R_L} \quad \text{and} \quad \varepsilon_2 = \frac{I_{inj2}}{I_1} = \frac{R_L}{R}$$

For weak coupling, a large value of R would be considered. Hence, for $R \gg R_L$ the parallel R in the model can be neglected and the model would be simplified to that presented in Fig. 9.

In our simulations, first we have used 4 oscillators in linear and circular configuration and then 6 oscillators in these configurations. The measurement scheme for the locking was given as follows. For a given coupling coefficient, the oscillators are divided in two equal groups. One group had the free-running frequency of a value of

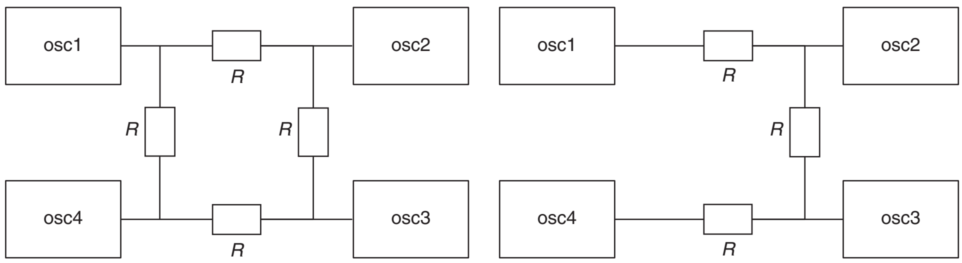

Fig. 7 *Establishment of coupling between oscillators in linear and circular arrangements*

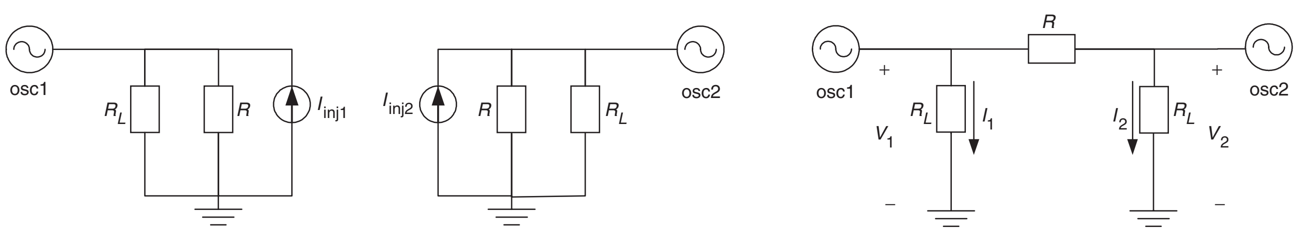

Fig. 8 *Equivalent circuit for coupling network*

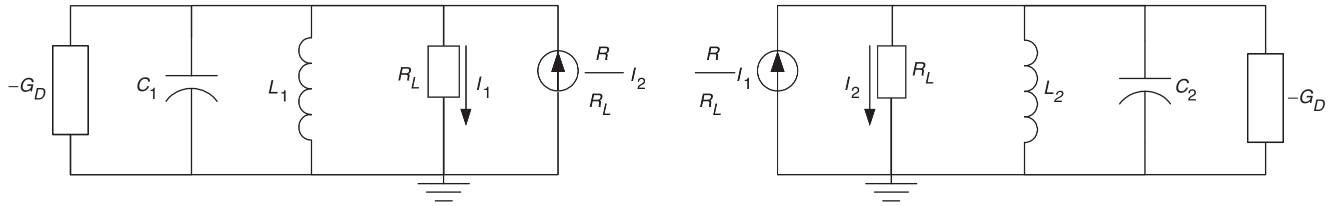

Fig. 9 *Simplified model for coupling network*

$f_0 + \Delta f$ (higher than the average), and the other group for which the free-running frequency were adjusted to a value of $f_0 - \Delta f$ (lower than the average frequency). In the experiment the Δf value is increased until the array unlocks, Δf_{max} is determined consequently. The typical output spectrums of locked and unlocked oscillators are depicted in Figs. 10 and 11.

Figure 12 shows the variations of locking bandwidth as a function of coupling coefficient for the four element linear and circular arrays resulted from ADS simulation. As seen in the Figure, the locking bandwidth for the circular array is roughly twice as much as that of the linear array. The inverse variations with the number of elements can also be seen in the Fig. 13, as the locking bandwidth for the four element array is roughly 50% more than that of six element array, predicted by (26).

An array of four element oscillators has been constructed and locking bandwidth was measured for linear and circular arrangements. The coupling resistor was chosen to be 4.7 kΩ which results in a coupling coefficient $\varepsilon = 0.213$. Typical measured locked and unlocked spectrums for this array are depicted in Figs. 14 and 15. The measured locking bandwidth for the linear and circular array were 234 kHz and 401 kHz, respectively, this shows the locking bandwidth for the circular array was about 1.71 times that of the linear array. The measured locking bandwidths were somehow

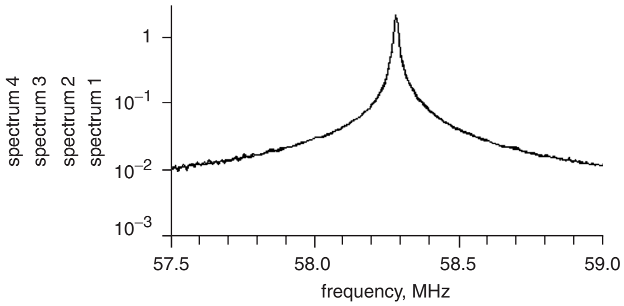

Fig. 10 *Typical spectrum of locked oscillators in an array with 4 cells and $\Omega_0 = 27$ kHz, $\varepsilon = 0.1$ (ADS simulation)*

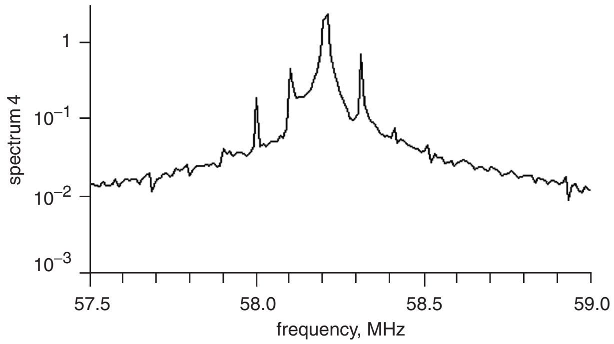

Fig. 11 *Spectrum of oscillator 4 on the verge of unlocking $\Omega_0 = 125$ kHz, $\varepsilon = 0.1$ (ADS simulation)*

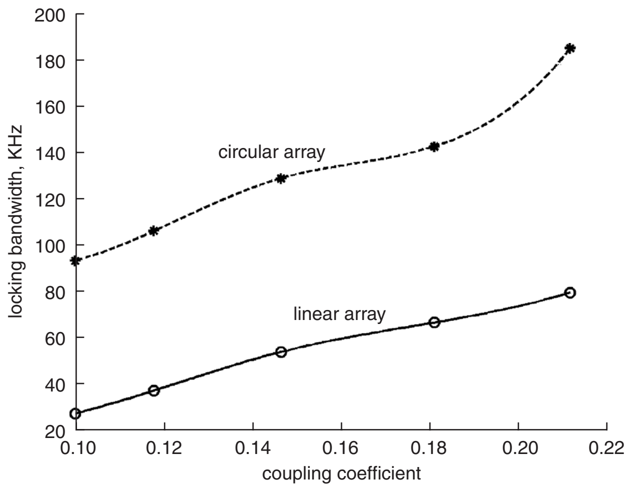

Fig. 12 *Variations of locking bandwidth as a function of coupling coefficient for four element linear and circular array of oscillators obtained using ADS simulation*

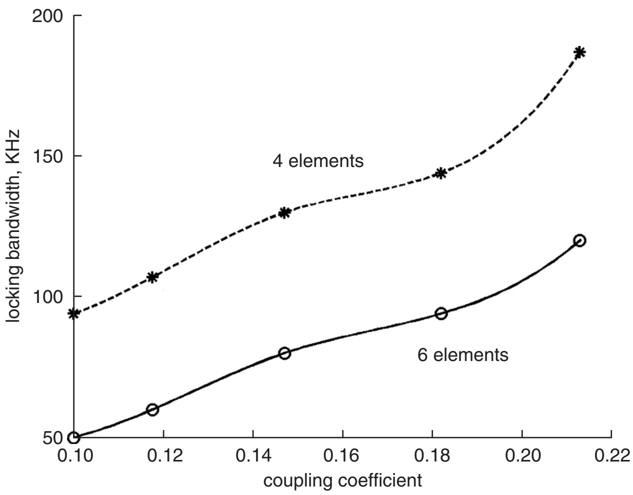

Fig. 13 *Comparison of locking bandwidth as a function of coupling factor for four and six element circular arrangements using ADS simulation*

more than theoretical ones in the constructed arrays specially for the linear array. This could be explained by the fact that, first, the real oscillators are not exactly similar in terms of output power or oscillation level, and, secondly, they were not completely isolated on the circuit board, consequently some additional coupling between the oscillators existed. We could say there exists some minor coupling through the circuit board and also through radiation, such that the minor extra coupling increases the locking bandwidth. For the linear array, this minor coupling causes its locking bandwidth to become a little bit larger than half of that of the circular array.

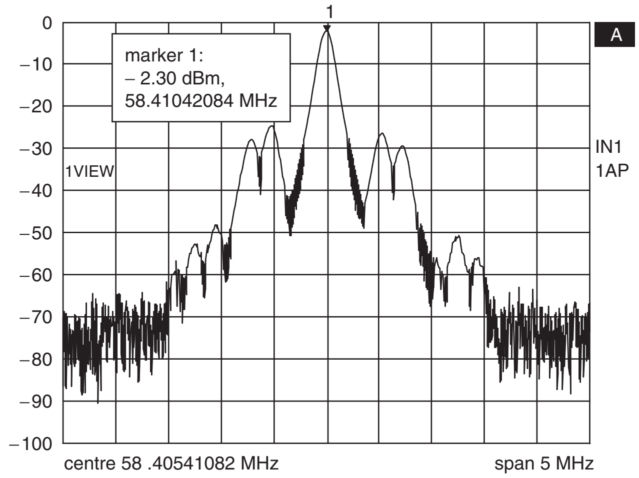

Fig. 14 *Typical measured unlocked spectrum of fabricated oscillator array (linear arrangement)*

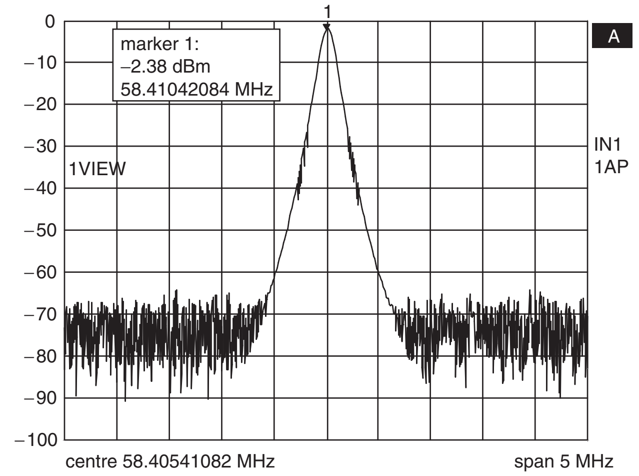

Fig. 15 *Typical measured spectrum of fabricated oscillator array after locking (linear arrangement)*

6 Conclusion

The locking bandwidth of an array of mutually coupled oscillators in linear and circular configurations was studied analytically. The locking bandwidth was defined and calculated in each case showing the circular array's locking bandwidth being twice as large as that of a linear array according to that definition. The results were validated computationally by a nonlinear circuit simulation program and experimentally by fabricating an array of VHF oscillators whose free-running frequencies and couplings could be changed practically through tuning capacitors and coupling resistors. More than double locking bandwidth was observed in circular array with respect to linear array in simulations. While the practical oscillator arrays had a minor increase in the locking bandwidth, due to parasitic coupling, with respect to theoretical prediction, an increase of 71% in the locking bandwidth in the circular array was observed practically compared with the linear array.

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