Compensating for Distortions in Interpolation of Two-Dimensional Signals Using Improved Iterative Techniques

A. ParandehGheibi, M. A. Rahimian, M. A. Akhaee, A. Ayremlou, F. Marvasti[1]

Advanced Communications Research Institute
Sharif University of Technology
Tehran, Iran
[1]marvasti@sharif.edu

Abstract— **In this paper we extended a previously investigated modular method that is designed to compensate for interpolation distortions of one-dimensional signals, to two dimensions (2-D). Next the proposed 2-D modular technique was applied in an iterative fashion and was shown through both simulations and theoretical analyses to enhance the convergence of the iterative technique. In fact, with only a few modules we were able to achieve drastic improvements in signal reconstruction, and with a much less computational complexity. Moreover, both the simulations and the theoretical analysis confirmed the robustness of the proposed scheme against additive noise.**

Keywords-iterative approach; interpolation distortion; quadrate latice; modular method; signal reconstruction; image interpolation;

I. INTRODUCTION

Signal reconstruction is often a necessary part of practical communication and information technology applications. Interpolators such as linear or cubic Splines are commonly used in all such applications. These interpolators, however, always incur some distortions at the Nyquist rate after low pass filtering. Reference [1] introduces an iterative method that can be exploited to alleviate the distortion caused by the reconstruction process. Reference [2] proposes a modular method to compensate for interpolation distortion of one-dimensional signals. The iterative method outperforms the modular method at the cost of more computations. In this paper we use an extended version of the modular method in an iterative fashion to enhance the convergence rate and distortion cancellation in the reconstruction of 2-D signals.

The rest of this paper is organized as follows: in the next section we briefly describe the modular and iterative methods and the extension of the former to two dimensions. Next in section III, we propose a novel scheme that exploits the extended modular technique in an iterative fashion to compensate for the interpolation distortions of 2-D signals. The performance of the proposed method is simulated in section IV using both 2-D random signals and real images. Section V concludes this paper.

II. PRELIMINARIES

A. Modular Method in 1-D

This method compensates the distortion of common interpolators such as Sample and Hold (S&H) and linear order hold by mixing the sum of cosine waves and then passing them through a lowpass filter.

If $s(t)$ is the interpolation of $x(t)$ samples, according to [1] we have:

$$\hat{x}_s(t) = s(t)\left[1 + 2\cos\left(\frac{2\pi t}{T}\right) + \cdots + 2\cos\left(\frac{2N\pi t}{T}\right)\right], \quad (1)$$

$$\hat{x}(t) = LPF\{\hat{x}_s(t)\}. \quad (2)$$

As N increases, $\hat{x}_s(t)$ converges to the ideal samples of $x(t)$ and thus, $\hat{x}(t)$ converges to $x(t)$.

B. Modular Method in 2-D

We can extend the 1-D Modular method to 2-D signals. For example for S&H we have:

$$h_{S\&H} = \sum_k \sum_{k'} \delta(t_x - kT)\delta(t_y - kT) \star \left(\prod\left(\frac{t_x}{T}\right)\prod\left(\frac{t_y}{T}\right)\right). \quad (3)$$

Thus in the 2-D case, the distortion function can be interpreted as:

$$sinc(f_x T).\, sinc(f_y T) \prod_{x,y}\left(\frac{f_{x,y}}{T}\right), \quad (4)$$

in which $\prod_{x,y}\left(\frac{f_{x,y}}{T}\right)$ is a rectangular surface represented as an ideal 2-D Lowpass Filter (LPF). In order to compensate the distortion function, we can add up $sinc$ functions in 2-D. Although there are different scenarios depending on the sampling scheme, we just focus on rectangular lattice structure which is common in sampling theory. In this case, as illustrated in Fig. 1, each sample is located in the lattice point. Therefore, the ideal LPF can be obtained by:

$$\sum_{k,k' \in Lattic\ point} \sum sinc(f_x T - k')sinc(f_y T - k) \equiv 1. \quad (5)$$

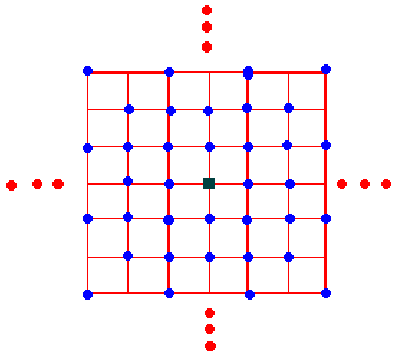

Figure 1. The Quadrature lattice

In the time domain, we have:

$$x_s(t) = s(x,y)\left[1 + 2\cos\left(\frac{2\pi x}{T_x}\right) + 2\cos\left(\frac{2\pi y}{T_y}\right) + 4\cos\left(\frac{2\pi x}{T_x}\right)\cos\left(\frac{2\pi y}{T_y}\right) + \cdots\right]. \quad (6)$$

It is obvious that as the number of modules increases, the results will converge to its ideal value. We shall see that with a few number of modules we can get good results.

C. Iterative Method

The iterative method to compensate for interpolation distortion is given by:

$$x_{k+1}(t) = \lambda Gx(t) + (I - \lambda G)x_k(t), \quad (7)$$

where λ is the relaxation parameter that determines the convergence rate and $x_k(t)$ is the k-th iteration. In addition, Operator G consists of two operators; P is a band-limiting operator and S is a sampling process, e.g., S can be S&H or LI and P can be a lowpass filter.

III. PROPOSED HYBRID APPROACH

One of the main disadvantages of the traditional iterative method is its low convergence rate, even for the optimum relaxation parameter. Since the modular method outperforms simple low-pass filtering, it can be exploited to improve the convergence rate of the iterative method. In order to combine the modular and iterative methods, we incorporate the modular method in each iteration step as shown in Fig. 2.

We will see in the simulation section that with only one module a phenomenal improvement can be achieved. Below, we shall prove the convergence for the S&H interpolation. The proof for other types of interpolation functions is similar.

Proof of Convergence for the S&H Interpolation:

For the P and S operators defined for *S&H*, we can write

$$x_{k+1}(t) = \lambda PSx(t) + x_k(t) - \lambda PSx_k(t), \quad (8)$$

where

$$Sx(t) = \left[1 + 2\cos\left(\frac{2\pi t}{T}\right) + \cdots\right]\sum_n x(nT)\prod\left(\frac{t-n}{T} - \frac{1}{2}\right). \quad (9)$$

$\prod(\frac{t}{T})$ is a rectangular function used for S&H. $x_k(t)$ will converge to $x(t)$ in the limit if we have a contraction, i.e., $\|I - \lambda G\| < 1$. This implies [2]:

$$\|x_{k+1} - x_k\| < \|x_k - x_{k-1}\|. \quad (10)$$

Substituting (8) in (10), we can get:

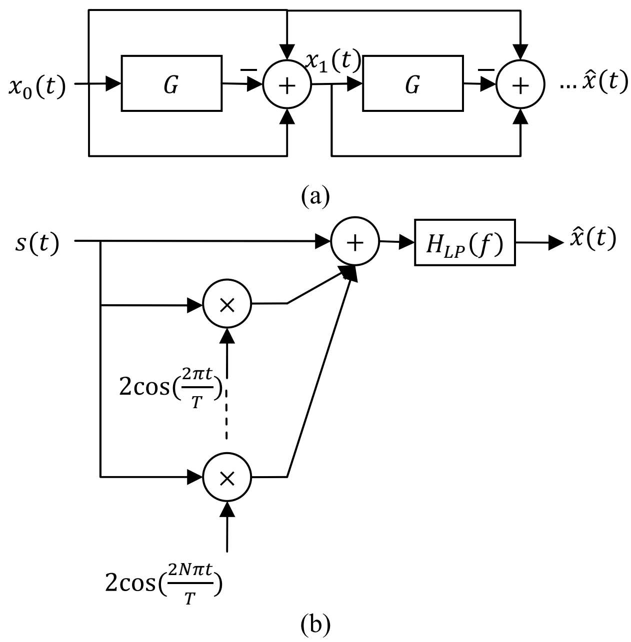

Figure 2. a) The reconstruction block diagram using standard iterative method b) The Modular Method

$$\|x_k - x_{k-1} - \lambda Gx_k - \lambda Gx_{k-1}\| \leq \|I - \lambda G\|\|x_k - x_{k-1}\| = r\|x_k - x_{k-1}\|, \quad (11)$$

where $G = PS$ and $0 \leq r = \sup\|I - \lambda G\| < 1$.

Assuming that only one module is applied, the left-hand side of (11) can be rewritten in the frequency domain as follows:

$$\left\|X_k(f) - X_{k-1}(f) - \lambda\prod([sinc(fT) + sinc(fT-1) + sinc(fT+1)]) \times \sum_k\left\{X_k\left(f - \frac{i}{T}\right) - X_{k-1}\left(f - \frac{i}{T}\right)\right\}\right\|, \quad (12)$$

where $\prod(fT)$ is an ideal lowpass filter with the cut-off frequency of $f_c = \frac{1}{2T}$. Assuming that the sampling rate is at the Nyquist rate, (12) becomes:

$$\|[X_k(f) - X_{k-1}(f)].\{1 - \lambda[sinc(fT) + sinc(fT-1) + sinc(fT+1)]\}\|. \quad (13)$$

Hence,

$$\|X_k(f) - X_{k-1}(f)\| \leq \max|1 - \lambda[sinc(fT) + sinc(fT-1) + sinc(fT+1)]|\,.\,\|X_k(f) - X_{k-1}(f)\|. \quad (14)$$

To satisfy (14), it is required that

$$r = \max|1 - \lambda[sinc(fT) + sinc(fT-1) + sinc(fT+1)]| < 1. \quad (15)$$

The maximum occurs at $f = \frac{1}{2T}$ and for $\lambda = 1$, and we get:

$$r = \left|1 - sinc\left(\frac{1}{2}\right) - sinc\left(\frac{-1}{2}\right) - sinc\left(\frac{3}{2}\right)\right| = 0.06 < 1. \quad (16)$$

Therefore, the proposed hybrid method converges to the original signal. From (11), assuming that G is an ideal low pass filter, r can be computed as $r = 1 - sinc\left(\frac{1}{2}\right) = 0.36$. Comparing this value with $r = 0.06$ derived for the hybrid method; we expect a drastic convergence rate improvement. For the best convergence rate, the relaxation parameter λ should be chosen so that it minimizes r, thus at the Nyquist rate, the optimal value for λ is given by:

$$\lambda_{opt} = \frac{1}{sinc\left(\frac{1}{2}\right) - sinc\left(\frac{-1}{2}\right) - sinc\left(\frac{3}{2}\right)} \cong 0.94. \quad (17)$$

For other types of interpolations, the derivations are similar. For example, for LI we have:

$$0 < r = \max\left|1 - \lambda[sinc^2(fT) + sinc^2(fT-1) + sinc^2(fT+1)]\right| < 1. \quad (18)$$

We get $r = 0.234 < 1$ for $\lambda = 1$, which is less than $r = 0.59$ for the conventional linear interpolation. The optimum value for λ is then given by:

$$\lambda_{opt} = \frac{1}{sinc^2\left(\frac{1}{2}\right) + sinc^2\left(-\frac{1}{2}\right) + sinc^2\left(\frac{3}{2}\right)} \cong 1.31. \quad (19)$$

If we mix the signal with more harmonics in each iteration step, as shown in Fig. 2(b), it can be easily shown that r decreases as we increase the number of modules. In the limit we can write:

$$\sum_{k=-\infty}^{\infty} sinc(fT - k) = 1. \quad (20)$$

Hence, r tends to zero for $\lambda = 1$ as the number of harmonics increases, and thus a faster convergence rate is expected.

For the proposed method in 2-D, the rectangular lattice sampling process with interpolation functions such as S&H and linear order hold were considered. The band limiting process P is an ideal 2-D LPF. All the relations regarding the convergence rate can be restated for the 2-D case. Since the procedure and the mathematical proof closely follow those of the 1-D case, we avoid rewriting them.

In the following subsections we will investigate the effect of noise and sampling rate on the convergence of the iterative technique, and then briefly describe the computational advantages brought about through incorporation of the modules in the iterative technique. Later in section IV, we use simulations to supplement the theoretical results derived in this section.

A. Noise Analysis

Suppose that the proposed hybrid method is used in a noisy environment. For the sake of analysis, white Gaussian noise is added to the original signal before the reconstruction. In this section we will analyze and compare the effects of noise on hybrid and traditional methods. From (8), for the traditional iterative method, we have:

$$x_{k+1}(t) = \lambda P x_s(t) + x_k(t) - \lambda P x_{sk}(t) + (\lambda P)^k n(t), \quad (21)$$

where $n(t)$ is the additive white Gaussian noise to the input, and $x_s(t)$ and $x_{sk}(t)$ are the S&H versions of $x(t)$ and $x_k(t)$, respectively. The necessary constraint on the convergence is the contraction inequality given in (11). Substituting (21) in (10), we obtain

$$\|x_k - x_{k-1} - \lambda G(x_k - x_{k-1}) + ((\lambda P)^k - (\lambda P)^{k-1})n\| \leq r\|x_k - x_{k-1}\|. \quad (22)$$

By invoking the triangle inequality, it is sufficient to have

$$\|x_k - x_{k-1} - \lambda G(x_k - x_{k-1})\| + \|(\lambda P)^k - (\lambda P)^{k-1}\|\|n\| \leq r\|x_k - x_{k-1}\|. \quad (23)$$

If we have a contraction then

$$\|(\lambda P)^k - (\lambda P)^{k-1}\|\|n\| \leq \|\lambda P\|^{k-1}\|n\|. \quad (24)$$

As in the previous section, the following inequality

$$\|X_k - X_{k-1}\|(r - |1 - \lambda sinc(fT)|_{\max}) \geq \|\lambda P\|^{k-1}\|n\|, \quad (25)$$

in the frequency domain, should be satisfied for $0 \leq r < 1$. For the worst case we have:

$$\|n\| \leq \frac{1}{\lambda^{k-1}}\|X_k - X_{k-1}\|\left(\lambda sinc\left(\frac{1}{2}\right)\right) \cong 0.318\lambda^{2-k}\|X_k - X_{k-1}\|. \quad (26)$$

This implies that as long as the noise standard deviation satisfies (26), the iteration will converge. Now, consider the proposed hybrid method. As in (25) we can state that

$$\|X_k - X_{k-1}\|(r - |1 - \lambda[sinc(fT-1) + sinc(fT) + sinc(fT+1)]|_{\max}) \geq \lambda^{k-1}\|n\|, \quad (27)$$

is a sufficient constraint for the convergence. And for the worst case, we have

$$\|n\| \leq \frac{1}{\lambda^{k-1}}\left(r - \left|-\lambda\left[sinc\left(-\frac{1}{2}\right) + sinc\left(\frac{1}{2}\right) + sinc\left(\frac{3}{2}\right)\right]\right|_{\max}\right) \cong 0.531\lambda^{2-k}\|X_k - X_{k-1}\|. \quad (28)$$

Comparing (26) with (28), we conclude that the proposed hybrid method can tolerate more noise power.

B. Sampling Rate Analysis

In the previous sections, the analysis was based on the sampling rate at the Nyquist rate. Suppose the sampling process is k times the Nyquist rate. Invoking the sufficient condition for convergence (11), we have

$$r = \left|1 - sinc\left(\frac{1}{2k}\right) - sinc\left(\frac{1}{2k} - 1\right) - sinc\left(\frac{1}{2k} + 1\right)\right| < 1. \quad (29)$$

For example, for $k = 2$, r is equal to 0.02. This factor is about 3 times smaller than that of the Nyquist rate; this implies we should expect $10\log(9) = 9.5$ dB improvement in terms of SNR at each iteration step. Similar analysis shows that an equivalent improvement for the LI can be expected. Although all the relations were proved for ordinary iterative method, the extension of the proof to the hybrid method and 2-D signals is straightforward.

C. Computational Complexity

The major advantage of the proposed hybrid method is its higher rate of convergence with less overall computational complexity. The conventional iterative method requires $M(4\log(2N) + 2)$ real additions and $M(2\log(2N) + 1)$ real multiplications per sample, where M is the number of iterations and N is the FFT block size. The hybrid method with one module, on the other hand, requires $M(4\log(2N) + 4)$ additions and $M(2\log(2N) + 3)$ multiplications per sample.

As for the 2-D case, since each of the above computations is performed in one dimension (per each row) and then the same is repeated in the other (per each column), the overall computational complexity is the same as the 1-D case but multiplied by $2K$, where K is the size of the 2-D square matrix. Although the number of computations for the hybrid case in each step of iteration is more, with a fewer number of iterations it achieves the same results and thus its overall computational load is considerably less.

In the next section, we simulate the performance of the proposed algorithm using both random 2-D signals and real images.

IV. SIMULATION RESULTS AND DISCUSSION

Fig. 3 shows the performance of the hybrid approach with different number of modules versus the number of iterations. The performance criterion for our simulations is the Signal to Noise Ratio (SNR) in dB. To have a fair comparison, initial band-limited signals are produced by FFT lowpass filtering of a Gaussian process with zero mean and an average power of 34 dB. During all simulations, we use the same FFT lowpass filter. The performance of each method is averaged over 50 signals. Here, λ is set to one, and sampling is at Nyquist rate. To avoid transient errors at the end points, SNR is calculated for interior points and 10% of the end points are ignored.

As it can be observed, the number of modules directly affects the quality of reconstruction. When a small number of modules are used, the iterative method attains a better performance. However, as the number of modules is increased, the role of the iterative method becomes less significant.

A. Noisy Environment

To simulate the performance of the algorithm in a noisy environment Additive White Gaussian Noise (AWGN), with a power of -20 dB, were added to the initial band-limited signals. This is a model of the channel noise that enters the reconstruction module along with the signal. Fig. 4 shows the results. As depicted, the curve of SNR improvement in a noisy environment saturates sooner than the noiseless environment. Nevertheless, the proposed method outperforms the traditional iterative method. It is evident from this figure that the more the number of modules, the better the reconstruction. Moreover, it can be deduced that the proposed algorithm enjoys a greater degree of robustness compared to the simple iterative approach.

B. Effcet of the Relaxation Parameter

In order to evaluate the optimum value of λ, we calculated the SNR improvement during 5 iterations in the proposed hybrid approach with different number of modules. Fig. 5 demonstrates the average SNR improvement versus different values of λ for 0, 1, 2 and 3 modules. Based on the figure, It can be inferred that by increasing the number of modules, the algorithm becomes nearly independent of λ. Even without any iteration (the starting point), we have a good reconstruction and the iterative algorithm just attains minor increase in SNR. Thus, the role of the iterative algorithm and its relaxation parameter λ decreases with the increasing number of modules.

On the other hand, when the iterative algorithm has a significant role, the value of λ and its effect on the SNR improvement become noticeable. From Fig. 5, for the simple iterative method (without modules), the optimum λ is 1.15, while using modules, this value tends to one.

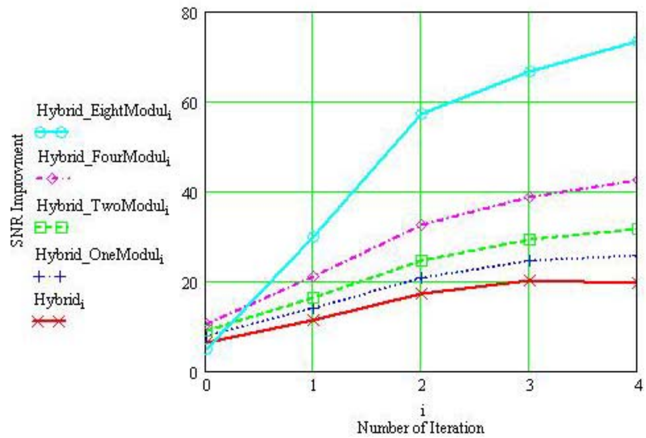

Figure 3. The performance of the hybrid approach with different number of modules versus the number of iterations (2-D, at Nyquist rate).

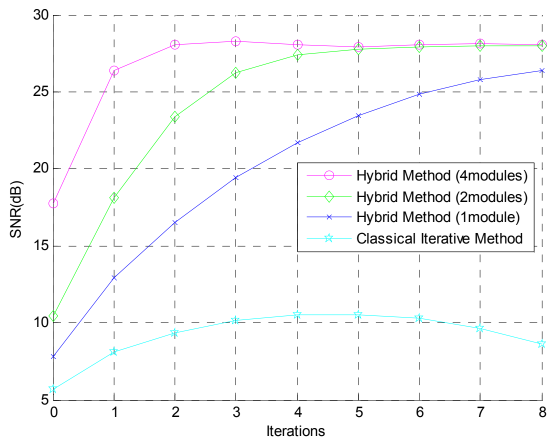

Figure 4. The SNR improvement versus the number of iterationsin the presence of noise, for different methods and with different number of modules, (2-D, the initial S /N = 27.69dB).

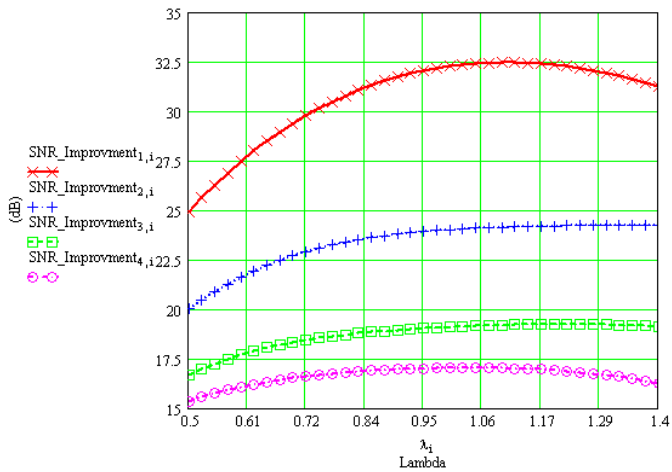

Figure 5. Average SNR improvements for hybrid approach with 0 (index $i = 1$), 1($i = 2$), 2($i = 3$) and 3($i = 4$) modules vs. different λ (2-D at Nyquist rate).

C. Application to Real Images

In the end, to evaluate the performance of the proposed method subjectively, we apply our technique to a well-known image (*Lena)*. In fact, by this algorithm, we intend to increase the size of the images with acceptable quality.

The errors between the high-resolution originals and reconstructed images are expressed in terms of PSNR (Peak Signal to Noise Ratio) values. Table I shows the errors for 4× enlargement. Objective comparisons based on PSNR are carried out with conventional bilinear and cubic Spline interpolation (we confirmed the results of [6]) as well as state-of-the art wavelet based methods [3]-[6]. A non-wavelet scheme based on edge-directed interpolation [7] was also considered to provide a comparison with an established method not operating in the wavelet domain. Our results show that the proposed method outperforms the other methods. Besides, the Hybrid method with only 1 module and 2 iterations exhibits almost the same PSNR performance as the classical iterative method with 10 iterations.

Fig. 6 shows the result of subjective comparison with bicubic Spline interpolation for the image *Lena*. Notice the sharpness of the *Lena* image enlarged with the proposed method in Fig. 6 (mid. Left) compared to the bicubic method (mid. Right), especially around lips, hat, and eyes. Overall, the hybrid method yields images that are sharper than the Spline or iterative method. Furthermore, as it is depicted in Fig. 6, the enlarged image by the proposed method (bottom Left) has lower errors around edges than other one (bottom Right).

V. CONCLUSION

A combination of the modular and hybrid methods was proposed for the compensation of interpolation distortion in 2-D signals. The superior performance of the proposed scheme was confirmed through both simulations and theoretical analysis. The major advantage of the proposed hybrid method is its higher rate of convergence with less overall computational complexity. Although the number of computations for the hybrid case in each step of iteration is more, with a fewer number of iterations it achieves the same results and thus its overall computational load is considerably less. In the future we plan to focus on the application of the hybrid method to 2-D signals, where hexagonal sampling has been used.

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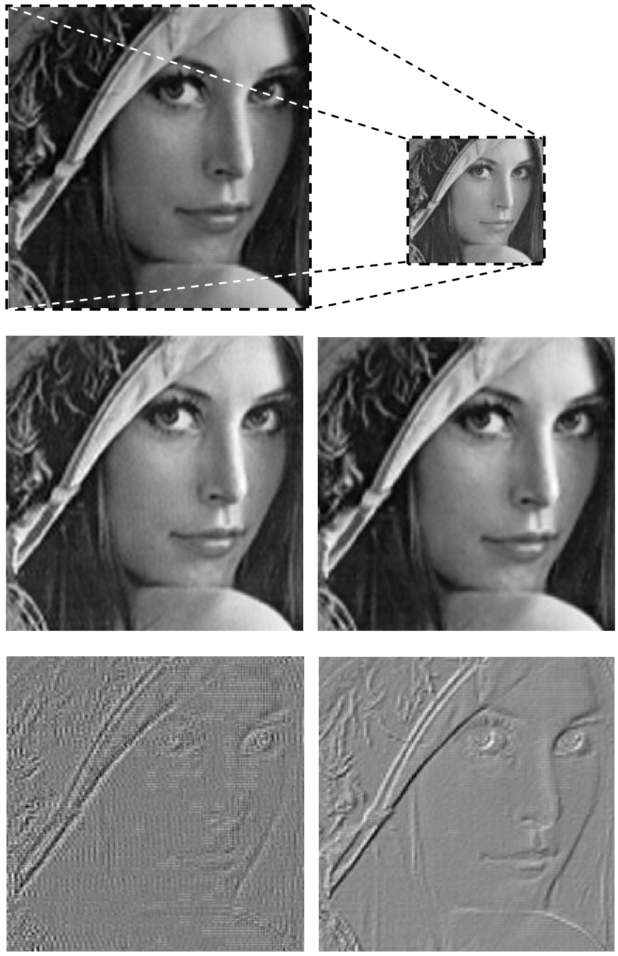

Figure 6. Top Left: Extract from original *Lena* image, Top Right: Original Image reduced by 4, Mid Right: 4× reconstruction using bicubic Spline interpolation, Mid Left: 4× reconstruction using hybrid method with 2 iterations and 1 module, Bottom right: Error corresponding to the bicubic interpolation and Bottom Left: Error corresponding to the hybrid method.

TABLE I
PSNR (dB) RESULTS FOR 4× ENLARGEMENT IMAGES
(FROM 256×256 TO 512×512)

Method	*PSNR (dB)*
Bilinear[6]	30.13
Bicubic[6]	31.34
NEDI [6-7]	34.10
WZP (Haar)[6]	31.46
WZP (Db.9/7)[6]	34.45
Carey et al. [3],[6]	34.48
HMM [4],[6]	34.52
HMM SR [5],[6]	34.61
WZP and CS [6]	34.93
Iterative (2 iter.)[1]	**35.25**
Iterative (10 iter.)[1]	**37.39**
Proposed Hybrid (2 iter. And 1 mod.)	**37.12**

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