Code-Division Multiple-Access Techniques in Optical Fiber Networks—Part III: Optical AND Logic Gate Receiver Structure With Generalized Optical Orthogonal Codes

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***Abstract*—In this paper, we present a deep insight into the behavior of optical code-division multiple-access (OCDMA) systems based on an incoherent, intensity encoding/decoding technique using a well-known class of codes, namely, optical orthogonal codes (OOCs). As opposed to parts I and II of this paper, where OOCs with cross-correlation $\lambda = 1$ were considered, we consider generalized OOCs with $1 \leq \lambda \leq w$, where w is the weight of the corresponding codes. To enhance the performance of such systems, we propose the use of an optical AND logic gate receiver, which, in an ideal case, e.g., in the absence of any noise source, except the optical multiple-access interference, is optimum. Using some basic laws on probability, we present direct and exact solutions for OOCs with $\lambda = 1, 2, 3, \ldots, w$, with the optical AND logic gate as receiver. Using the exact solution, we obtain empirical solutions that can be easily used in optimizing the design criteria of such systems. From our optimization scheme, we obtain some fresh insight into the performance of OOCs with $\lambda \geq 1$. In particular, we can obtain some simple relations between $P_{e\min}$ (minimum error rate), $L_{\min}$ (minimum required OOC length), and $N_{\max}$ (maximum number of interfering users to be supported), which are the most desired parameters for any OCDMA system design. Furthermore, we show that in most practical cases, OOCs with $\lambda = 2, 3$ perform better than OOCs with $\lambda = 1$, while having a much bigger cardinality. Finally, we show that an upper bound on the maximum weight of OOCs are on the order of $\sqrt{2\lambda L}$ where L is the length of the OOCs.**

***Index Terms*—Generalized OOCs, optical AND logic gate receiver structure, optical code-division multiple-access (OCDMA), optical orthogonal codes (OOCs), optimum auto- and cross-correlation value, optimum weight.**

I. Introduction

EVER SINCE the introduction of fundamental principles of optical code-division multiple-access (OCDMA) using on–off pulses as signature sequences, the search for powerful code structures that can simultaneously satisfy three conditions, namely, minimal auto- (λ_a) and cross- (λ_c) correlation values, while having large cardinality, was unleashed [1]–[3]. Among the most famous codes introduced to date are optical orthogonal codes (OOCs with its variations), and prime sequences [4], [5].

OOCs, which have captured the attention of many mathematicians and sequence designers [3]–[5], enjoy a more superior characteristic than prime sequences, because of their more robust definition and conditions. By definition, a family of OOCs can be designed for any code length (L), code weight (w), autocorrelation (λ_a), and cross-correlation (λ_c) values, while for a given code length (L), prime sequences are designed with weights (w) that are on the order of the square root of its length ($\sqrt{L}$), and with cross-correlation values that are bounded by two ($\lambda_c = 2$) and autocorrelation values that can take on any value between one to code weight ($1 \leq \lambda_a \leq w$), thereby making them more limited in their use.

From a multiaccess and synchronization point of view, the most desirable on–off signature sequences are OOCs with $\lambda_a = \lambda_c = 1$. However, these families of codes may suffer from low cardinality in some applications. In [6]–[8], attempts were made to explore the performance of OCDMA systems that employ OOCs with $1 \leq \lambda_c \leq w$. In particular, in [7], by obtaining lower and upper bounds on the system's performance, it was hinted that OOCs with $\lambda_c = 2$ could, under certain conditions, outperform OCDMA employing OOCs with $\lambda_c = 1$, with cardinality that could be a hundred to a thousand times bigger. Similarly, in [9], using a structure based on mimicking a framed time-hopping ultra-wideband CDMA system, it was shown that the system's performance degrades gradually with respect to OOCs with $1 \leq \lambda_c \leq w$, while offering many possible signature sequences.

In this paper, we first obtain a simple solution on the performance of OCDMA systems using generalized OOCs, i.e., OOCs with $\lambda_c = 2, 3, \ldots$, or w, and then using the analytical solution, we obtain the best code parameter values and find the most powerful codes for three various design scenarios in an OCDMA system. We show that for most practical purposes, OOCs with $1 \leq \lambda_c \leq 3$ achieve the best performance, and we find the corresponding optimum weights that achieve the best performance.

These results resemble those of prime sequences, but with major differences in their autocorrelation value and their cardinality. OOCs with $\lambda_c = 1, 2, 3$ could be designed to have $\lambda_a \leq 3$, as opposed to prime sequences autocorrelation value that can take on any value between 1 and w. Furthermore, the number of codes for OOCs with $\lambda_c = 2, 3$ is on the order of $\approx (L^{\lambda_c}/w^{\lambda_c+1})$, while for prime sequences, is on the order of

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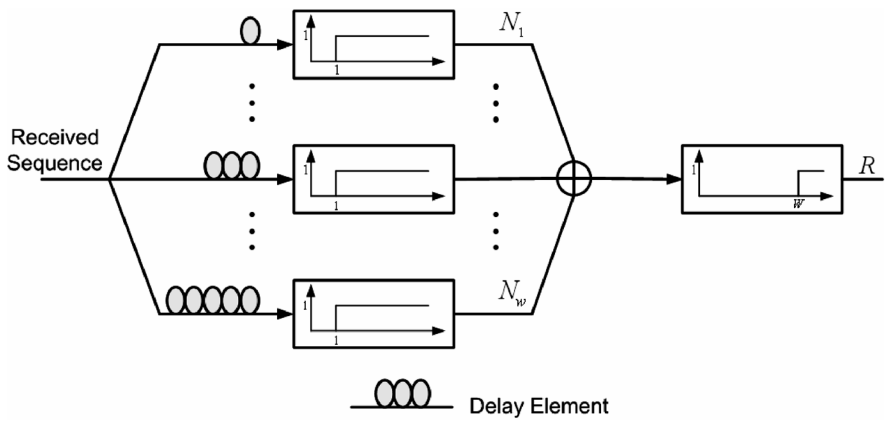

Fig. 1. Ideal optical AND logic gate receiver structure for all-optical code recognition.

L. Hence, for sufficiently large L, i.e., $L > w^{(\lambda_c+1)/(\lambda_c-1)}$, there are many more OOCs than prime sequences.

The rest of this paper is organized as follows. In Section II, we explain our system model. In Section III, we analyze the performance for an OCDMA system with OOCs having the corresponding cross-correlation values $\lambda = 1, 2$, and $\lambda \geq 1$. Section IV discusses some properties of generalized OOCs. In Section V, we explain in a precise language our design problem, and describe optimum design criteria, pointing out the limitation on OCDMA systems using generalized OOCs. In Section VI, we obtain a simple empirical solution for our design criteria that proves to be extremely useful. In Section VII, we conclude this paper.

II. System Model

The system considered in this paper is the common on–off keying (OOK) OCDMA with optical modulators and optical fiber-delay-element-based encoders and decoders. $N+1$ is the number of active users, and as a consequence, we have $N+1$ OOK modulators and optical encoders that convert the *on* bits to signature sequences of length L and weight w, and *off* bits to signature sequences of length L and weight zero. For the sake of notational simplicity, we assume that N is the number of interfering users. Our multiple-access channel is an asynchronous incoherent additive channel. However, without loss of generality and for the sake of mathematical simplicity, we assume that even in the absence of any bit synchronism between any two users, we have chip-synchronism among them. Surely, we can safely assume that the performance based on the chip-synchronism model will be an upper bound on the actual chip and bit asynchronism [1], [2]. For our mathematical model, we use $N+1$ independent identically distributed random variables j_i, each with uniform distribution on the integers $\{1, 2, \ldots, L\}$, which determines the random start time of the ith user with respect to a common reference. Since each transmitter–receiver pair is assumed to be perfectly synchronized, we can assume that j_i can be determined exactly for the ith transmitter–receiver pair, but it remains random to other users' transmitter–receiver pairs.

In this paper, we use generalized OOCs, which assume the maximum cross-correlation between any two optical signature sequences to be not greater than $\lambda (\lambda \in \{1, 2, \ldots, w\})$. From the above discussion, for the remainder of this paper, we assume that L, w, λ, N are fixed and known parameters of our OCDMA system design.

Our receiver structure is based on an optical AND logic gate, which is, in fact, an all-optical implementation of a chip-level receiver [10]–[13], Fig. 1, with their thresholds set at Th. This implies that for a given input value x, output y is determined by the rule [2]

$$y = \begin{cases} 1, & x \geq \mathrm{Th} \\ 0, & x < \mathrm{Th}. \end{cases} \tag{1}$$

In this paper, we assume $\mathrm{Th} = 1$ or w, depending on the position of the optical hard-limiter used within the optical AND logic gate receiver structure, as discussed in the following. A receiver structure based on an optical AND logic gate not only gives a better performance in our system, but also, for small values of N, it can be shown that optimum receiver reduces to an optical AND logic gate followed by a correlator [11]. Besides, performance evaluations of a receiver structure solely based on an optical correlator can be found easily, and we do not discuss them further. In our bit-asynchronous chip-synchronous OCDMA model, all users are mutually independent and send their corresponding binary data with equal probability, and the ith receiver behaves and decides as follows [10].

1) With the help of j_i, the receiver determines the pulse positions, with a normalized intensity equal to one constituting its desired optical code in the received sequence, which is the sum of $N+1$ signature codes of different users.
2) Constructs w random variables, namely, N_l with $l = 1, 2, \ldots, w$ that can take on two values, namely, zero or one, which are the output of w hard-limiters, each tuned to the lth pulse position of its corresponding signature code.
3) Decides on on–off data with the following rule:

$$R = \begin{cases} 1, & N_1 + N_2 + \cdots + N_w = w \\ 0, & N_1 + N_2 + \cdots + N_w < w. \end{cases} \tag{2}$$

The above receiver structure is equivalent to an optical AND logic gate that can be represented by the following alternative and equivalent behavior and decision rule (see Fig. 2).

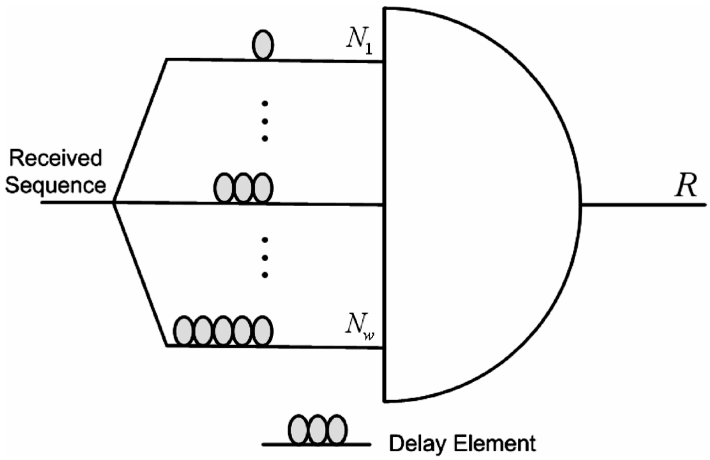

Fig. 2. Equivalent functional AND logic gate receiver structure.

1) With the help of j_i, the receiver determines the pulse positions in the desired receive sequence.
2) Constructs w random variables, namely, N_l with $l = 1, 2, \ldots, w$, that can take on any values between zero and $N+1$, which are the values of the lth chip pulse position in the desired receive sequence.
3) Decides on on–off data with the following rule:

$$R = \begin{cases} 1, & N_1 N_2 \cdots N_w > 0 \\ 0, & N_1 N_2 \cdots N_w = 0. \end{cases} \quad (3)$$

Without any loss of generality, and for the sake of mathematical simplicity, we use the above equivalent representation on the receiver structure to evaluate the performance of our proposed OCDMA, based on a generalized OOCs system using an equivalent optical AND logic gate at their receiver structure. Furthermore, we note that our performance analysis in this paper does not take into account any sources of noise other than optical multiple-access noise, such as background noise, shot or quantum noise, and thermal noise.

Following the visualization method introduced in part I [1], in obtaining the output of auto- and cross-correlation values via optical disk-pattern concepts, Fig. 3 demonstrates the equivalent operation of the optical AND logic gate using the optical disk-pattern concept. In optical AND logic gate operation, the interfering codes are first combined among themselves, then matched with the desired user's OOCs, as opposed to adding the interfering OOCs, then matching it with the desired user's OOCs in a simple correlation structure. If we have $N+1$ OOCs, i.e., $C_0, C_1, \ldots, C_N$ with C_0 denoting the desired code, then the equivalent mathematical operation for the optical AND logic gate is $C_0 \cdot \max(C_1, C_2, \ldots, C_N)$, where for a simple correlator receiver is $C_0 \cdot (C_1 + C_2 + \cdots + C_N)$. Since the elements of OOCs take on values of 0 or 1, the output of the elements of an optical AND logic gate will remain 0 or 1, whereas the output elements of the correlator could take on any values between 0 and N. In Fig. 3, one can observe the interference combination effect, i.e., the multiplication effect of the optical AND logic gate, by noting that when one pulse of each interfering user are overlapped, and subsequently, those pulses are overlapped with the pulse position of the matched filter, we only observe one pulse with intensity equal to one on the projection surface, rather than a pulse with intensity equal to two for simple correlator structure. One can conclude that optical AND logic gate operation eliminates all those interfering patterns which reach the receiver threshold by analog summation of light intensities, rather than reconstructing (up to the threshold) the exact replica of the desired OOC.

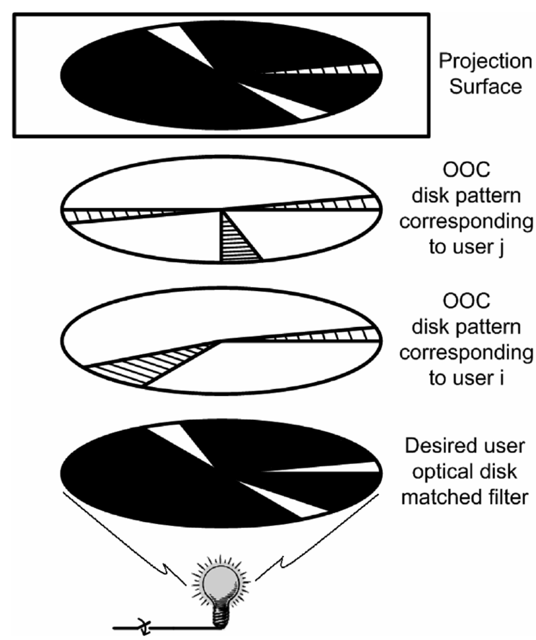

Fig. 3. Demonstrating and visualizing optical AND logic gate operation on the cross-correlation value of two interfering users upon the desired user's matched filter, using the concept of optical disk patterns. If we denote matched filter by A and interfering i and j codes by B and C, respectively, then the mathematical equivalent of the above optical AND logic gate demonstration is $A \cdot \max(B, C)$, as opposed to $A \cdot (B + C)$ for a simple correlator receiver.

III. Performance Analysis

A. Performance Analysis for OOCs With $\lambda = 1$

For a given desired user (for example, the first user), we can write

$$P_e = \frac{1}{2}\Pr(\text{error}|0) + \frac{1}{2}\Pr(\text{ error}|1). \quad (4)$$

However, since the system is positive and additive, the probability of error conditioned on transmitting bit "1" is zero [2]; i.e.,

$$\begin{aligned}\Pr(\text{error}|1) &= \Pr(R = 0|1)\\ &= \Pr(N_1 N_2 \cdots N_w = 0|1) = 0. \quad (5)\end{aligned}$$

Therefore, in what follows, we focus on determining the probability of error when the desired user sends bit "0." In this case, we assume that we have N independent interfering users, where each may or may not interfere with one marked chip position of the first user with equal probability. Therefore, if the ith user interferes, we may define $a_i^{(1)}$ as its interfering position. Superscript "1" in the definition of $a_i^{(1)}$ is due to the fact that each interfering user only interferes with one chip position of the desired user. Hence, $a_i^{(1)}$ is a random variable with the following property:

$$\begin{cases}\Pr\left(a_i^{(1)} = 1\right) = \frac{w}{2L}\\ \Pr\left(a_i^{(1)} = 2\right) = \frac{w}{2L}\\ \vdots\\ \Pr\left(a_i^{(1)} = w\right) = \frac{w}{2L}\end{cases} \quad (6)$$

where parameter $(w/2L)$ is the probability of a hit of the ith interfering user with an OOC length L and weight w, with specified chip positions. Furthermore, if we denote by p_0 the probability of no hit in any chip pulse position of the intended user by the ith interfering user, then it can be deduced easily that $p_0 = 1 - w \times (w/2L) = 1 - (w^2/2L)$. Following the definition of (6) and p_0, the total probability distribution for random variable $a_i^{(1)}$ is

$$\begin{aligned}\Pr\left(a_i^{(1)}\right) &= \left(1 - \frac{w^2}{2L}\right)\delta\left(a_i^{(1)}\right)\\ &+\frac{w}{2L}\delta\left(a_i^{(1)} - 1\right) + \cdots + \frac{w}{2L}\delta\left(a_i^{(1)} - w\right). \quad (7)\end{aligned}$$

Hence, from (7), we note that N_l is the total number of users that satisfy $a_f^{(1)} = l$, and thus, has a binomial distribution as follows:

$$\begin{aligned}\Pr(N_1, N_2, \ldots, N_w) &= \frac{N!}{N_1! N_2! \cdots N_w! \left(N - \sum_{l=1}^{w} N_l\right)!}\\ &\times\left(\frac{w}{2L}\right)^{\sum_{l=1}^{w} N_l}\left(1 - \frac{w^2}{2L}\right)^{N - \sum_{l=1}^{w} N_l}. \quad (8)\end{aligned}$$

However, since $\Pr(\text{error}|0) = \Pr(R = 1|0) = 1 - \Pr(R = 0|0)$, we can write

$$\begin{aligned}\Pr(R = 0|0)&\\ &= \Pr(N_1 N_2 \cdots N_w = 0|0)\\ &= \Pr[(N_1 = 0) \text{ or } (N_2 = 0) \cdots \text{ or } (N_w = 0)|0]\\ &= \sum_{l=1}^{w}\Pr(N_l = 0) - \frac{1}{2!}\sum_{\substack{l,l'=1\\ l\neq l'}}^{w}\Pr(N_l = 0, N_{l'} = 0)\\ &+ \frac{1}{3!}\sum_{\substack{l,l',l''=1\\ l\neq l'\neq l''}}^{w}\Pr(N_l = 0, N_{l'} = 0, N_{l''} = 0) - \cdots. \quad (9)\end{aligned}$$

In the above relation, we use the well-known property of calculating the probability of the union of events with the aid of the probability of their mutually exclusive events. But with the help of (6) and (7), we have

$$\Pr(N_1 = 0, N_2 = 0, \ldots, N_k = 0) = \left(1 - \frac{kw}{2L}\right)^N. \quad (10)$$

Hence, by using (10) in (9), we have

$$\begin{aligned}\Pr(R = 0|0)&\\ &= w \times \left(1 - \frac{w}{2L}\right)^N - \frac{w(w-1)}{2!} \times \left(1 - \frac{2w}{2L}\right)^N\\ &+ \frac{w(w-1)(w-2)}{3!} \times \left(1 - \frac{3w}{2L}\right)^N - \cdots\\ &= \sum_{k=1}^{w}(-1)^{k+1}\binom{w}{k}\left(1 - \frac{kw}{2L}\right)^N. \quad (11)\end{aligned}$$

Thus, (4) can be written as

$$\begin{aligned}P_e &= \frac{1}{2}\Pr(\text{error}|0) = \frac{1}{2}[1 - \Pr(R = 0|0)]\\ &= \frac{1}{2}\left[1 + \sum_{k=1}^{w}(-1)^k\binom{w}{k}\left(1 - \frac{kw}{2L}\right)^N\right]. \quad (12)\end{aligned}$$

The above equation is equivalent to the previously obtained results in [7], [8], [12], [14], and [15], using completely different approaches. This surprising and very simple argument motivates us to use this method for $\lambda > 1$.

B. Performance Analysis for OOCs With $\lambda = 2$

In Section III-A, we introduced our method for simplest possible case, namely, OOCs with $\lambda = 1$. In this part, we extend our method in obtaining probability of error for OOCs with $\lambda = 2$. And finally, in Section III-C, we will further extend our method to the most general case, namely, OOCs with $1 \leq \lambda \leq w$.

For $\lambda = 2$, (6) must be modified so as to contain all possible different cases. To attain this goal, we should rewrite (6) as

$$\begin{cases}\Pr\left(a_i^{(1)} = 1\right) = p_1\\ \Pr\left(a_i^{(1)} = 2\right) = p_1\\ \vdots\\ \Pr\left(a_i^{(1)} = w\right) = p_1\end{cases} \quad (13\text{-a})$$

$$\begin{cases} \Pr\left(a_i^{(2)}=(1,2)\right)=p_2 \\ \Pr\left(a_i^{(2)}=(1,3)\right)=p_2 \\ \vdots \\ \Pr\left(a_i^{(2)}=(w-1,w)\right)=p_2 \end{cases} \tag{13-b}$$

where $a_i^{(1)}$ in (13-a) corresponds to a hit on one pulse position, and $a_i^{(2)}$ in (13-b) corresponds to hits on two pulse positions. In the above relations, we consider all possible interference patterns caused by the ith interfering user and their corresponding probabilities, where p_1 is the probability that one interfering user hits on one chip pulse position of the intended user's OOC. Similarly, p_2 is the probability that it hits on two chip pulse positions of the intended user's OOC, and as such, we define p_0 to be the probability of no hit on any chip pulse position of the intended user by the interfering user. It can be expressed as follows:

$$p_0 = 1 - p_1 \times w - p_2 \times \frac{w(w-1)}{2!}. \tag{14}$$

However, we note that if we consider one interfering user, its total probability of interfering is $(w^2/2L)$, since there are only w^2 chip pulse positions that the interfering user can hit on L possible chip positions, with its data being *on* or *off* with probability $(1/2)$. Furthermore, the total probability of interference due to p_1 and p_2 can be written as $1\times p_1\times w+2\times p_2\times(w(w-1)/2!)$, thus it follows that

$$1\times p_1 \times w + 2\times p_2 \times \frac{w(w-1)}{2!} = \frac{w^2}{2L}. \tag{15}$$

For further analysis, we define new random variables, namely, N'_l for $l\in\{1,2,\ldots,w\}$, and $N'_{ll'}$ for $l\neq l'\in\{1,2,\ldots,w\}$. N'_l is the total number of users that satisfy $a_f^{(1)}=l$, and similarly, $N'_{ll'}$ is the total number of $a_f^{(2)}$'s that satisfy $a_f^{(2)}=(l,l')$. Note that $N'_{ll'}=N'_{l'l}$. Obviously, and similar to Section III-A, these random variables have multinomial distribution as given in (16), shown at the bottom of the page.

On the other hand, the random variables $N_1,N_2,\ldots,N_w$ are related to $N'_1,\ldots,N'_w,N'_{12},\ldots,N'_{(w-1)w}$ through w linear equations, as follows:

$$\begin{cases} N_1 = N'_1 + N'_{12} + N'_{13} + \cdots + N'_{1w} \\ N_2 = N'_2 + N'_{21} + N'_{23} + \cdots + N'_{2w} \\ \vdots \\ N_w = N'_w + N'_{w1} + N'_{w2} + \cdots + N'_{w(w-1)}. \end{cases} \tag{17}$$

Even though obtaining $\Pr(N_1,N_2,\ldots,N_w)$ could prove to be mathematically tedious, based on our formulation, we do not need to obtain the above expression directly.

As we discussed before, $\Pr(\text{error}|1)=0$, and hence, $P_e=(1/2)\Pr(\text{error}|0)$, thus (9) is still valid and (10) is modified to [see (17)]

$$\begin{aligned}\Pr(N_1=0,N_2=0,\ldots,N_k=0) &= \Pr(N'_1=0,\ldots,N'_k=0,N'_{12}=0,\ldots,N'_{1w}\\ &=0,\ldots,N'_{k1}=0,\ldots,N'_{\mathrm{kw}}=0).\end{aligned} \tag{18}$$

For a multinomial distribution, calculating for the above expression reduces to the following simple form:

$$\begin{aligned}\Pr(N_1=0,N_2=0,\ldots,N_k=0)\\ = [1-kp_1-((w-1)+(w-2)+\cdots+(w-k))p_2]^N\end{aligned} \tag{19}$$

hence

$$\begin{aligned}P_e = \frac{1}{2}+\frac{1}{2}\sum_{k=1}^{w}(-1)^k\binom{w}{k}\\ \times[1-kp_1-((w-1)+(w-2)+\cdots+(w-k))p_2]^N.\end{aligned} \tag{20}$$

C. Performance Analysis for Generalized OOCs

Consider a code set with an arbitrary value of $1\leq\lambda\leq w$. Following the same argument and methodology as in Sections III-A and III-B, and without any further discussion, we can extend (14), (15), and (19) as follows:

$$\sum_{k=1}^{\lambda}\binom{w}{k}p_k = 1-p_0 \tag{21}$$

$$\sum_{k=1}^{\lambda}k\binom{w}{k}p_k = \frac{w^2}{2L} \tag{22}$$

$$\begin{aligned}&\Pr(N_1=0,\ldots,N_k=0)\\ &=\Big[1-k\frac{p_1}{0!}-[(w-1)+\cdots+(w-k)]\frac{p_2}{1!}\\ &\quad-[(w-1)(w-2)+\cdots+(w-k)(w-k-1)]\frac{p_3}{2!}\cdots\\ &\quad-[(w-1)(w-2)\cdots(w-\lambda+1)+\cdots+(w-k)\\ &\quad\cdot(w-k-1)\cdots(w-k-\lambda+2)]\frac{p_\lambda}{(\lambda-1)!}\Big]^N.\end{aligned} \tag{23}$$

$$\begin{aligned}&\Pr(N'_1,\ldots,N'_w,N'_{12},\ldots,N'_{(w-1)w})\\ &=\frac{N!}{N'_1!\cdots N'_w!N'_{12}!\cdots N'_{(w-1)w}!\left(N-\sum_{l=1}^{w}N'_l-\sum_{\substack{l,l'=1\\ l<l'}}^{w}N'_{ll'}\right)!}p_1^{\sum_{l=1}^{w}N'_l}\times p_2^{\sum_{l,l'=1,l<l'}^{w}N'_{ll'}}\times p_0^{N-\sum_{l=1}^{w}N'_l-\sum_{l,l'=1,l<l'}^{w}N'_{ll'}}\end{aligned} \tag{16}$$

However, it can be shown that (see Appendix A)

$$\Pr(N_1=0,N_2=0,\ldots,N_k=0)=\left[1-\sum_{n=1}^{\lambda}\frac{wp'_n}{n}\times\left(1-\frac{(w-n)(w-n-1)\cdots(w-n-k+1)}{w(w-1)\cdots(w-k+1)}\right)\right]^N \quad (24)$$

where we have defined (for all $1\leq n\leq\lambda$)

$$p'_n=p_n\binom{w-1}{n-1}. \quad (25)$$

Now, directly we can write

$$P_e=\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{w}(-1)^k\binom{w}{k}\left[1-\sum_{n=1}^{\lambda}\frac{wp'_n}{n}\times\left(1-\frac{(w-n)(w-n-1)\cdots(w-n-k+1)}{w(w-1)\cdots(w-k+1)}\right)\right]^N. \quad (26)$$

It is interesting to note that the above result is similar to a result obtained previously by Chen and Yang, using an elegant approach based on Markov chain rules [8].

For example, if our codes are designed so that only $p_\lambda\neq 0$ and all other values of p_k are zero, then from (22) and (25), we have $p'_\lambda=(w/2L)$ and $p'_k=0$ for all $1\leq k\leq\lambda-1$, and thus (26) reduces to

$$P_e=\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{w}(-1)^k\binom{w}{k}\left[1-\frac{w^2}{2\lambda L}\times\left(1-\frac{(w-\lambda)(w-\lambda-1)\cdots(w-\lambda-k+1)}{w(w-1)\cdots(w-k+1)}\right)\right]^N. \quad (27)$$

IV. Limits on the Performance of a Typical OCDMA Using Optical and Logic Gate Receiver Structure With Generalized OOCs

A. A Fundamental Theorem

In this section, we state an important and fundamental theorem that specifies a crucial property for various classes of OOCs used in OCDMA. At first, and for future use, let us rewrite (21) and (22) versus p'_k as

$$\sum_{k=1}^{\lambda}\frac{w}{k}p'_k=1-p_0 \quad (28)$$

$$\sum_{k=1}^{\lambda}p'_k=\frac{w}{2L}. \quad (29)$$

Assume $1\leq i<j\leq w$, and assume $p'=[p'_1,p'_2,\ldots,p'_w]$, with some p'_k's possibly zero, to be a valid vector in assigning some codes. Now, consider two elements p'_i and p'_j, and also assume that all other elements of p' are constant.

Theorem 1: P_e is a monotonically increasing function of p'_j on $[0,p'_{j\,\max}]$, i.e., $(\partial P_e/\partial p'_j)\geq 0$, for

$$p'_{j\,\max}=\frac{w}{2L}-\sum_{\substack{n=1\\n\neq i,j}}^{w}p'_n. \quad (30)$$

It is obvious [see (29)] that any increase on p'_j should be balanced with an equivalent decrease on p'_i, and we use this fact in our calculations. For the proof, see Appendix B.

Result 1: If our codes are designed such that

$$\begin{cases}p'_k\neq 0, & 1\leq\lambda_1\leq k\leq\lambda_2\leq w\\ p'_k=0, & \text{otherwise}\end{cases} \quad (31)$$

then for lower and upper bounds on P_e, we obtain the results in (32), shown at the bottom of the page.

Proof is directly achieved with the help of (28), (29), and *Theorem 1*.

B. An Upper Bound on the Maximum Value of w

Equation (28) gives us an important result. For any code set, following the restriction $0\leq p_0\leq 1$, we must have

$$\sum_{k=1}^{\lambda}\frac{p'_k}{k}\leq\frac{1}{w}\Rightarrow w\leq\left(\sum_{k=1}^{\lambda}\frac{p'_k}{k}\right)^{-1}. \quad (33)$$

The important equation above tells us that we can not increase w without any bound. For an example, if our codes are designed such that $p'_\lambda=(w/2L)$ and $p'_k=0$ for all $1\leq k\leq\lambda-1$ then $w\leq\sqrt{2\lambda L}$.

$$P_e\geq\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{w}(-1)^k\binom{w}{k}\left[1-\frac{w^2}{2\lambda_1 L}\left(1-\frac{(w-\lambda_1)(w-\lambda_1-1)\cdots(w-\lambda_1-k+1)}{w(w-1)\cdots(w-k+1)}\right)\right]^N$$

$$P_e\leq\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{w}(-1)^k\binom{w}{k}\left[1-\frac{w^2}{2\lambda_2 L}\left(1-\frac{(w-\lambda_2)(w-\lambda_2-1)\cdots(w-\lambda_2-k+1)}{w(w-1)\cdots(w-k+1)}\right)\right]^N \quad (32)$$

C. An Alternative Lower Bound on P_e

Besides *Theorem 1*, we can directly find another lower bound for P_e. To show this, it is sufficient to use (17) and note that

$$\Pr(1 \le N'_1, 1 \le N'_2, \ldots, 1 \le N'_w) \le \Pr(1 \le N_1, 1 \le N_2, \ldots, 1 \le N_w). \quad (34)$$

But

$$\begin{aligned}\Pr(1 \le N'_1, 1 \le N'_2, \ldots, 1 \le N'_w) &= 1 - \Pr[(N'_1 = 0) \text{ or } (N'_2 = 0) \text{ or } \cdots \text{ or } (N'_w = 0)] \\ &= 1 - \sum_{l=1}^{w} \Pr(N'_l = 0) + \frac{1}{2!} \sum_{\substack{l,l'=1 \\ l \ne l'}}^{w} \Pr(N'_l = 0, N'_{l'} = 0) \\ &\quad - \frac{1}{3!} \sum_{\substack{l,l',l''=1 \\ l \ne l' \ne l''}}^{w} \Pr(N'_l = 0, N'_{l'} = 0, N'_{l''} = 0) + \cdots \\ &= 1 + \sum_{k=1}^{w} (-1)^k \binom{w}{k} [1 - kp_1]^N \end{aligned} \quad (35)$$

hence

$$P_e \ge \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{w} (-1)^k \binom{w}{k} [1 - kp_1]^N. \quad (36)$$

But (34) can be written for larger sets, and the above argument for those situations is also true. For example, we can write

$$\Pr(1 \le N'_1, \ldots, 1 \le N'_w, 1 \le N'_{12}, \ldots, 1 \le N'_{(w-1)w}) \le \Pr(1 \le N_1, 1 \le N_2, \ldots, 1 \le N_w) \quad (37)$$

and immediately we can extend our proof to this case. Thus, in general, and for any $\chi \subseteq \{1, 2, \ldots, \lambda\}$, we have

$$P_e \ge \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{w} (-1)^k \binom{w}{k} \left[1 - \sum_{n \in \chi} \frac{w p'_n}{n} \times \left(1 - \frac{(w-n)(w-n-1)\cdots(w-n-k+1)}{w(w-1)\cdots(w-k+1)} \right) \right]^N. \quad (38)$$

D. Error-Free Communication via Generalized OOCs

Whenever we use generalized OOCs, $P_e = 0$ if and only if $\lambda N + 1 \le w$.

Proof: Consider (26), and rewrite it in the form of

$$P_e = \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{w} (-1)^k \binom{w}{k} \left(1 + \sum_{n=1}^{\lambda} \gamma_n k^n \right)^N \quad (39)$$

in which γ_n's are some appropriate coefficients. Then, we have

$$\begin{aligned} P_e &= \frac{1}{2} \sum_{k=0}^{w} (-1)^k \binom{w}{k} \times \sum_{n_1, \ldots, n_\lambda} \frac{N!}{n_1! n_2! \cdots n_\lambda! (N - n_1 - n_2 - \cdots - n_\lambda)!} \times \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_\lambda^{n_\lambda} k^{(n_1 + 2n_2 + \cdots + \lambda n_\lambda)} \\ &= \frac{1}{2} \sum_{n_1, \ldots, n_\lambda} \frac{N!}{n_1! n_2! \cdots n_\lambda! (N - n_1 - n_2 - \cdots - n_\lambda)!} \times \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_\lambda^{n_\lambda} \sum_{k=0}^{w} (-1)^k \binom{w}{k} k^{(n_1 + 2n_2 + \cdots + \lambda n_\lambda)}. \end{aligned} \quad (40)$$

But we know

$$\sum_{k=0}^{w} (-1)^k \binom{w}{k} k^n = 0 \quad \text{for } n < w. \quad (41)$$

Hence, if in all situations $n_1 + 2n_2 + \cdots + \lambda n_\lambda < w$, then $P_e = 0$, but this condition is satisfied if and only if $\lambda N < w$, which proves the result.

V. Optimum Design Criteria

In this section, we state specifically our design problem, and in the next section, we obtain numerical solutions to our design problem with the help of relations given in previous sections. At first, the discussion in this section may come as a surprise, because *Theorem 1* states that among all codes and for any situation, the codes with $\lambda = 1$ are the best possible codes, and the best performance will be attained with these codes. But in this section, we show that there is another side to this calculation, and the conclusion from the above theorem is only half of the story.

From another point of view, to design an OCDMA system, we need to first find all codes with a special structure based on a predefined $(L, p' = [p'_1, p'_2, \ldots, p'_w], w, \lambda)$, then choose $N + 1$ codeword amongst them. However, the most stringent restriction we need to consider here is that the number of codes based on the Johnson bound [3], [7] is finite, and thus we can not increase N without any limit. This, in effect, is the other half of the story.

Clearly, from the above discussion, we conclude that previous relations are valid only for those values of N which are bounded by the Johnson bound, and are not applicable for those values of N beyond their corresponding Johnson bound. Hence, by considering the above restriction, *Theorem 1* by itself is not complete and applicable, and finding a real applicable solution needs further investigation. Hence, the question of optimum design is not as simple as it seems at first, and this process may well be a very interesting and challenging problem for a designer of an OCDMA system.

To begin with, let us choose the simplest possible codes, i.e., codes with $p' = (0, \ldots, 0, (w/2L))$, in order to simplify our analytical discussion. Furthermore, these codes that are simpler than their corresponding general codes can be analyzed with parameters (L, N, w, λ) only, and their performance is an upper bound to the performance of their corresponding general codes. Therefore, analyzing the above simplified codes not only allows us to focus on the main ideas behind our design criteria, but in addition, the performance obtained for these codes is an upper bound on the performance of our design, and we use this fact for the rest of this paper.

Any generalized OOC OCDMA system can be specified with parameters (P_e, L, N, w, λ). As a system designer, we are interested in the following problem: Given two parameters among (P_e, L, N), how can we find (w, λ) such that the optimum value on the third parameter can be achieved? With optimum value on P_e, we mean its minimum possible value on L, the minimum required value, and on N, its maximum value. With the above design goal in mind, we reintroduce the design problems as follows.

Problem 1: For specified values of (L, N), what are the values of $(w_{\text{opt}}, \lambda_{\text{opt}})$ to reach minimum probability of error, i.e., $P_{e\,\text{min}}$?

Problem 2: For specified values of (P_e, N), what are the values of $(w_{\text{opt}}, \lambda_{\text{opt}})$ for minimum code length, i.e., L_{min}?

Problem 3: For specified values of (P_e, L), what are the values of $(w_{\text{opt}}, \lambda_{\text{opt}})$ to obtain the maximum number of interfering users, i.e., N_{max}?

As we discussed in the above, to solve these problems, we need to specify precisely the number of codewords for which the above parameters are valid. Actually, any change in the number of appropriate codewords alters our optimization search space, and thus results in different values. In this paper, we consider only codes which have the maximum out-of-phase cross-correlation value λ, i.e., generalized OOCs. For these codes, the Johnson bound states that

$$(N+1) \le \frac{(L-1)(L-2)\cdots(L-\lambda)}{w(w-1)(w-2)\cdots(w-\lambda)}. \tag{42}$$

Note that the above relation is simply an upper bound on N, and whenever a better or a tighter bound is found, results should be modified accordingly.

Here, we also limit ourselves to the values of $1 \le \lambda \le 3$. This limitation actually is not very restrictive, because in almost all practical purposes for $\lambda \ge 3$, we would obtain such a large number of codewords that (42) does not affect our search region seriously, and hence, we could safely ignore it. Therefore, the search region for $\lambda \ge 3$ according to Section IV-B will solely depend upon the code length L, i.e.,

$$3 \le \lambda \le w \le \sqrt{2\lambda L}. \tag{43}$$

However, in this region, *Theorem 1* simply states that the optimum value for λ is 3, i.e., $\lambda_{\text{opt}} = 3$, and thus, we need not extend our search beyond this. The above property guarantees that $\lambda_{\text{opt}} \le 3$. Hence, in this paper, our search space on (w, λ) is confined to

$$\begin{aligned} &1 \le \lambda \le 3 \\ &\lambda \le w \le \sqrt{2\lambda L} \\ &w(w-1)(w-2)\cdots(w-\lambda) \le \frac{(L-1)(L-2)\cdots(L-\lambda)}{N+1}. \end{aligned} \tag{44}$$

With the above relations, we intend to find $(w_{\text{opt}}, \lambda_{\text{opt}})$ for previously stated design problems. In the next section, we will find empirical formulas for $(w_{\text{opt}}, \lambda_{\text{opt}})$ using numerical methodology.

VI. Figures and Examples

This section is devoted to figures and discussions which show the dependence of $(w_{\text{opt}}, \lambda_{\text{opt}})$ on the intended design problem. To obtain results, we use (27). Our calculations for the following figures are for fixed values of $\lambda = 1, 2, 3$ (see Fig. 4). To be consistent with our previous discussions, in each search, we first obtain λ_{opt} and then obtain w_{opt}. In some regions, we find $w_{\text{opt}} = 0$. This implies that there is no code with corresponding λ to satisfy our conditions.

To expand our discussion on optimum design criteria, we can deduce some simple approximate relations from figures using a curve-fitting procedure. For example, in (47), we find the best coefficients that relate bit-error rate $(\text{BER})_{\text{dB}}$, (L/N), (L/N^2), and $\log_{10} L$, the most important parameters, in a linear equation. These approximate relations, which are valid on a wide range of desired system parameters, i.e., $L \le 1000$ and $N \le 100$, give us further insight on the performance of different OOC structures, and thus, we can use them as a measure of the overall performance bounds on an OCDMA system using generalized OOCs.

We present these relations in a design procedure scheme, not only to give the answer to the design problems of the previous section, but also to give examples for a real design procedure. To do this, we first define $\text{BER}_{\text{dB}} = 10\log_{10} P_{e\,\text{min}}$.

Design Procedure:

Problem 1: For specified values of (L, N), what are the values for $(w_{\text{opt}}, \lambda_{\text{opt}})$ to reach $P_{e\,\text{min}}$?

Step 1) Find an approximate value for λ_{opt} from (L, N) as

$$\lambda_{\text{opt}}(L,N) \approx \begin{cases} 1, & L \le 5N \\ 2, & 5N < L \le 0.5N^2 \\ 3, & 0.5N^2 < L \le 3N^2 \\ 2, & 3N^2 < L \le 10N^2 \\ 1, & 10N^2 < L. \end{cases} \tag{45}$$

Step 2) Find an approximate value for w_{opt} from (L, N) and λ_{opt} as

$$\begin{aligned} w_{\text{opt}} &\approx \min\left\{N+1, \sqrt{\frac{L}{N+1}}\right\} \quad \lambda_{\text{opt}} = 1 \\ w_{\text{opt}} &\approx \min\left\{2N+1, \sqrt[3]{\frac{L^2}{N+1}},\right. \\ &\qquad \left. L \times (5\times 10^{-3})^{\frac{N-0.02L-10}{N-0.02L+10}}\right\} \quad \lambda_{\text{opt}} = 2 \\ w_{\text{opt}} &\approx \min\left\{3N+1, \sqrt[4]{\frac{L^3}{N+1}},\right. \\ &\qquad \left. L \times (5\times 10^{-3})^{\frac{N-0.02L-10}{N-0.02L+10}}, 2.4\sqrt{L}\right\} \quad \lambda_{\text{opt}} = 3. \end{aligned} \tag{46}$$

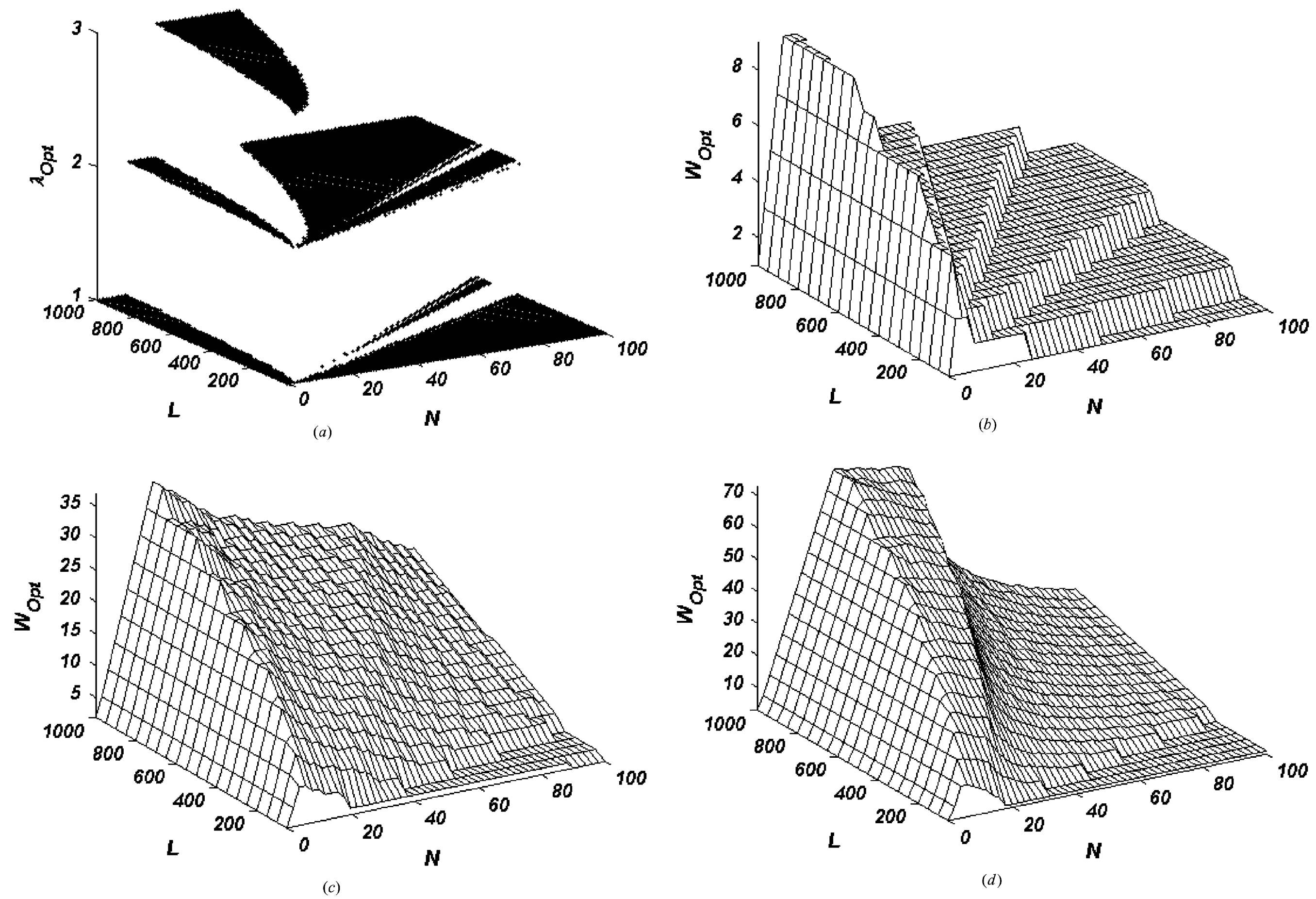

Fig. 4. $(w_{\rm opt}, \lambda_{\rm opt})$ for given (L, N). (a) $\lambda_{\rm opt}$. (b) $w_{\rm opt}$ for $\lambda = 1$. (c) $w_{\rm opt}$ for $\lambda = 2$. (d) $w_{\rm opt}$ for $\lambda = 3$.

Step 3) Find an approximate value for $P_{e\,\min}$ from (L, N) and $\lambda_{\rm opt}$ as

$$\begin{aligned} &\text{BER}_{\rm dB} + 1.5\frac{L}{N} + 1.4\frac{L}{N^2} + 6.6\log_{10} L \approx 5.1 \quad \lambda_{\rm opt} = 1\\ &\text{BER}_{\rm dB} + 3.6\frac{L}{N} + 2\frac{L}{N^2} - 2.2\log_{10} L \approx -5.6 \quad \lambda_{\rm opt} = 2\\ &\text{BER}_{\rm dB} + 3.3\frac{L}{N} + 19.4\frac{L}{N^2} - 4.9\log_{10} L \approx -8.6 \quad \lambda_{\rm opt} = 3. \end{aligned} \tag{47}$$

Problem 2: For specified values of (P_e, N), what are the values of $(w_{\rm opt}, \lambda_{\rm opt})$ in order to obtain the minimum value required on code length, i.e., $L_{\min}$?

Step 1) Find an approximate value for $\lambda_{\rm opt}$ from (P_e, N) as

$$\lambda_{\rm opt}(P_e, N) \approx \begin{cases} 2, & \text{BER}_{\rm dB} + 1.3N \geq -20\\ 3, & \text{otherwise.} \end{cases} \tag{48}$$

Step 2) Find an approximate value for $L_{\min}$ from (P_e, N) and $\lambda_{\rm opt}$ with the help of (47).

Step 3) Find an approximate value for $w_{\rm opt}$ from $(L_{\min}, N)$ and $\lambda_{\rm opt}$ with the help of (46).

Problem 3: For specified values of (P_e, L), what are the values of $(w_{\rm opt}, \lambda_{\rm opt})$ to reach $N_{\max}$?

Step 1) Find an approximate value for $\lambda_{\rm opt}$ from (P_e, L) as

$$\lambda_{\rm opt}(P_e, L) \approx \begin{cases} 2, & \text{BER}_{\rm dB} + 50\log_{10} L \geq 70\\ 3, & \text{otherwise.} \end{cases} \tag{49}$$

Step 2) Find an approximate value for $N_{\max}$ from (P_e, L) and $\lambda_{\rm opt}$ with the help of (47).

Step 3) Find an approximate value for $w_{\rm opt}$ from $(L, N_{\max})$ and $\lambda_{\rm opt}$ with the help of (46).

From our design procedure, we clearly observe that the optimum design criteria critically depends on the problem under consideration, i.e., obtaining $P_{e\,\min}, L_{\min}$, or $N_{\max}$. For example, if our main concern is the value of $N_{\max}$, we can achieve the best performance with values of $\lambda = 2, 3$ only. This should not be surprising, since OOCs with $\lambda = 1$ suffer seriously from low cardinality. On the other hand, if our goal is $P_{e\,\min}$, the case of $\lambda = 1$ may be optimum in certain conditions.

VII. Conclusion

In this paper, we presented a deep insight into the behaviors and characteristics of OCDMA based on OOCs with cross-correlation value $\lambda \geq 1$, and an optical AND logic gate structure as its receiver. The presentation into the behavior of OCDMA is based on obtaining the exact solution for the above-mentioned system. The exact solution presented here for OOCs with $\lambda \geq 1$

and optical AND logic gate receiver structures resolves, once and for all, a crucial analytical problem that existed in OCDMA. The most important result deduced from the exact solution is a set of empirical formulas that interrelate five important parameters, namely, minimum error rate $P_{e\min}$, minimum code length required, $L_{\min}$, maximum number of interfering users $N_{\max}$, optimum weight w_{opt}, and optimum cross-correlation value λ_{opt}. From our solution and numerical results, we observe that from practical point of view, OOCs with $\lambda = 2, 3$ are more desirable than OOCs with $\lambda = 1$, thereby giving us a strong hint into the importance of constructing and generating OOCs with $\lambda = 2, 3$.

Appendix A
Establishing (24)

We know

$$\binom{w-1}{n-1}+\binom{w-2}{n-1}+\cdots+\binom{w-k}{n-1} = \binom{w}{n}+\binom{w-k}{n}. \tag{A.1}$$

Hence we can write the equation shown in (A.2) at the bottom of the page. Now if we define (for all $1 \le n \le \lambda$)

$$p'_n = p_n \binom{w-1}{n-1} \tag{A.3}$$

then (23) reduces to

$$\Pr(N_1 = 0, N_2 = 0, \ldots, N_k = 0) = \left[1 - \sum_{n=1}^{\lambda} \frac{wp'_n}{n} \times \left(1 - \frac{(w-n)(w-n-1)\cdots(w-n-k+1)}{w(w-1)\cdots(w-k+1)}\right)\right]^N. \tag{A.4}$$

Appendix B
A Proof of *Theorem 1*

At first, we define

$$a_{w,k,n} = \frac{w}{n}\left(1 - \frac{(w-n)(w-n-1)\cdots(w-n-k+1)}{w(w-1)\cdots(w-k+1)}\right) \tag{B.1}$$

Now, if we are interested in $a_{w,k,n} - a_{w,k,n+1}$, then with help of (A.2), we can write

$$a_{w,k,n} - a_{w,k,n+1} = \frac{(w-n-1)!}{(w-1)!} \sum_{i=1}^{k} \frac{(i-1)\times(w-i)!}{(w-i-n+1)!}. \tag{B.2}$$

If we calculate the above equation for different values of n, after simple manipulations, we would see

$$\begin{aligned}
a_{w,k,1} - a_{w,k,2} &= \frac{1}{(w-1)} \sum_{i=1}^{k} (i-1) \\
&= \frac{k(k-1)}{2(w-1)} \\
a_{w,k,2} - a_{w,k,3} &= \frac{1}{(w-1)} \sum_{i=1}^{k} (i-1) \\
&\quad - \frac{1}{(w-1)(w-2)} \sum_{i=1}^{k} (i-1)(i-2) \\
&= \frac{k(k-1)}{2(w-1)} - \frac{k(k-1)(k-2)}{3(w-1)(w-2)} \\
a_{w,k,3} - a_{w,k,4} &= \frac{1}{(w-1)} \sum_{i=1}^{k} (i-1) \\
&\quad - \frac{2}{(w-1)(w-2)} \sum_{i=1}^{k} (i-1)(i-2) \\
&\quad + \frac{1}{(w-1)(w-2)(w-3)} \\
&\quad \times \sum_{i=1}^{k} (i-1)(i-2)(i-3) \\
&= \frac{k(k-1)}{2(w-1)} - \frac{2k(k-1)(k-2)}{3(w-1)(w-2)} \\
&\quad + \frac{k(k-1)(k-2)(k-3)}{4(w-1)(w-2)(w-3)} \\
&\ \vdots
\end{aligned} \tag{B.3}$$

The above relations suggest that we can write

$$\begin{aligned}
a_{w,k,i} - a_{w,k,j} &= \beta_1 \frac{k(k-1)}{(w-1)} - \beta_2 \frac{k(k-1)(k-2)}{(w-1)(w-2)} \\
&\quad + \beta_3 \frac{k(k-1)(k-2)(k-3)}{(w-1)(w-2)(w-3)} - \cdots \\
&\quad + (-1)^w \beta_{w-1} \frac{k(k-1)\cdots(k-w+1)}{(w-1)(w-2)\cdots 1}
\end{aligned} \tag{B.4}$$

$$\frac{(w-1)(w-2)\cdots(w-n+1) + \cdots + (w-k)(w-k-1)\cdots(w-k-n+2)}{(n-1)!} = \binom{w}{n}\left(1 - \frac{(w-n)(w-n-1)\cdots(w-n-k+1)}{w(w-1)\cdots(w-k+1)}\right) \tag{A.2}$$

in which β_m's for $1 \le m \le w-1$ satisfy $0 \le \beta_m$. To continue, we define

$$C_{w,k} = 1 - \sum_{n=1}^{\lambda} a_{w,k,n} p'_n. \tag{B.5}$$

With the help of (B.1) and (B.5), (26) reduces to

$$\begin{aligned} P_e &= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} \left[1 - \sum_{n=1}^{\lambda} a_{w,k,n} p'_n\right]^N \\ &= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} C_{w,k}^N. \end{aligned} \tag{B.6}$$

For our use, we rewrite (B.6) as

$$\begin{aligned} P_e &= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} \\ &\quad \times \left[1 - \sum_{\substack{n=1\\ n\ne i,j}}^{\lambda} a_{w,k,n} p'_n - a_{w,k,i} p'_i - a_{w,k,j} p'_j\right]^N. \end{aligned} \tag{B.7}$$

Now with the help of (29)

$$\begin{aligned} P_e &= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} \\ &\quad \times \left[1 - \sum_{\substack{n=1\\ n\ne i,j}}^{\lambda} a_{w,k,n} p'_n \right. \\ &\quad \left. - a_{w,k,i}\left(\frac{w}{2L} - p'_j - \sum_{\substack{n=1\\ n\ne i,j}}^{\lambda} p'_n\right) p'_i - a_{w,k,j} p'_j\right]^N \\ &= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} \\ &\quad \times \left[1 - \frac{w}{2L} \times a_{w,k,i} + \sum_{\substack{n=1\\ n\ne i,j}}^{\lambda} (a_{w,k,i} - a_{w,k,n}) p'_n \right. \\ &\quad \left. + (a_{w,k,i} - a_{w,k,j}) p'_j\right]^N. \end{aligned} \tag{B.8}$$

Thus, we can simply write

$$\begin{aligned} \frac{\partial P_e}{\partial p'_j} &= \frac{N}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} (a_{w,k,i} - a_{w,k,j}) \\ &\quad \times \left[1 - \frac{w}{2L} \times a_{w,k,i} + \sum_{\substack{n=1\\ n\ne i,j}}^{\lambda} (a_{w,k,i} - a_{w,k,n}) p'_n \right. \\ &\quad \left. + (a_{w,k,i} - a_{w,k,j}) p'_j\right]^{N-1} \\ &= \frac{N}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} (a_{w,k,i} - a_{w,k,j}) C_{w,k}^{N-1}. \end{aligned} \tag{B.9}$$

But from (B.4)

$$\begin{aligned} \frac{\partial P_e}{\partial p'_j} &= \frac{N\beta_1}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} \frac{k(k-1)}{(w-1)} C_{w,k}^{N-1} \\ &\quad - \frac{N\beta_2}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} \frac{k(k-1)(k-2)}{(w-1)(w-2)} C_{w,k}^{N-1} \\ &\quad + \frac{N\beta_3}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} \frac{k(k-1)(k-2)(k-3)}{(w-1)(w-2)(w-3)} C_{w,k}^{N-1} + \cdots \\ &\quad + (-1)^w \frac{N\beta_{w-1}}{2}\sum_{k=1}^{w} (-1)^k \binom{w}{k} \\ &\quad \times \frac{k(k-1)\cdots(k-w+1)}{(w-1)(w-2)\cdots 1} C_{w,k}^{N-1} \\ &= \frac{N\beta_1}{2}\sum_{k=2}^{w} (-1)^k \binom{w}{k} \frac{k(k-1)}{(w-1)} C_{w,k}^{N-1} \\ &\quad - \frac{N\beta_2}{2}\sum_{k=3}^{w} (-1)^k \binom{w}{k} \frac{k(k-1)(k-2)}{(w-1)(w-2)} C_{w,k}^{N-1} \\ &\quad + \frac{N\beta_3}{2}\sum_{k=4}^{w} (-1)^k \binom{w}{k} \frac{k(k-1)(k-2)(k-3)}{(w-1)(w-2)(w-3)} C_{w,k}^{N-1} + \cdots \\ &\quad + (-1)^w \frac{N\beta_{w-1}}{2} (-1)^w \binom{w}{w} \\ &\quad \times \frac{w(w-1)(w-2)\cdots 1}{(w-1)(w-2)\cdots 1} C_{w,w}^{N-1} \\ &= \frac{Nw\beta_1}{2}\sum_{k=2}^{w} (-1)^k \binom{w-2}{k-2} C_{w,k}^{N-1} \\ &\quad - \frac{Nw\beta_2}{2}\sum_{k=3}^{w} (-1)^k \binom{w-3}{k-3} C_{w,k}^{N-1} \\ &\quad + \frac{Nw\beta_3}{2}\sum_{k=4}^{w} (-1)^k \binom{w-4}{k-4} C_{w,k}^{N-1} + \cdots + \frac{Nw\beta_{w-1}}{2} C_{w,w}^{N-1} \\ &= \frac{Nw}{2}\left[\beta_1 \sum_{k=2}^{w} (-1)^{k-2} \binom{w-2}{k-2} C_{w,k}^{N-1} \right. \\ &\quad + \beta_2 \sum_{k=3}^{w} (-1)^{k-3} \binom{w-3}{k-3} C_{w,k}^{N-1} \\ &\quad \left. + \beta_3 \sum_{k=4}^{w} (-1)^{k-4} \binom{w-4}{k-4} C_{w,k}^{N-1} + \cdots + \beta_{w-1} C_{w,w}^{N-1}\right]. \end{aligned} \tag{B.10}$$

Clearly, each term in the above bracket is positive, and thus, on the entire valid range of $p'_j([0, p'_{j\,\max}])$, we have $0 \le (\partial P_e/\partial p'_j)$, and hence, the theorem is proved. It is important to note that $C = [C_{w,1}, C_{w,2}, \ldots, C_{w,w}]$ and all its powers $C^N = [C_{w,1}^N, C_{w,2}^N, \ldots, C_{w,w}^N]$ are valid vector probability distributions, such that when summed over, as in (B.10), it is still surely positive.

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