OPTIMIZED SPLINE INTERPOLATION

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ABSTRACT

The goal of this paper is to design compact support basis spline functions that best approximate a given filter (e.g., an ideal Lowpass filter). The optimum function is found by minimizing the least square problem (ℓ_2 norm of the difference between the desired and the approximated filters) by means of the calculus of variation; more precisely, the introduced splines give optimal filtering properties with respect to their time support interval. Both mathematical analysis and simulation results confirm the superiority of these splines.

Keywords— Spline, Interpolation, Filter Design

1. INTRODUCTION

The conversion of continuous-time signals such as multimedia data with discrete and digitized samples is a common trend nowadays. This is mainly due to the existence of powerful tools in the discrete domain. However, the conversion of continuous-time signals into the discrete form by means of sampling may destroy all or some parts of the data. Under certain conditions on the continuous signal, such as bandlimitedness [10], the sampling process is guaranteed to be one to one; i.e., there should be a priori a continuous model. In spite of the technological movement toward digital signal processing, by the advances in wavelet theory [2, 9, 11], a revival of continuous-time modeling for the digital data has been triggered. Multiresolution analysis [8, 18], self-similarity [3, 17], and singularity analysis [7] are inseparable from a continuous-time interpretation. It is therefore crucial to have efficient mathematical tools that allow easy switching from the digital domain to the continuous, and this is precisely the niche that splines, and, to some extent, wavelets, are trying to fill.

In this field, polynomial splines, such as B-splines, are particularly popular, mainly due to their simplicity, compact support, and excellent approximation capabilities compared other methods. Spline-based methods have spread to various applications since the development of B-splines [13, 14, 16].

Though B-splines generate remarkable results in many applications, they are not the optimum solutions for filtering problems such as interpolation. This paper, focuses on the problem of designing optimal compact support splines which best approximate a given filter such as the ideal lowpass filter. In fact, the desired filter reflects the characteristics of the continuous-time model and can be arbitrary.

The remainder of the paper is organized as follows: The next section briefly describes the spline interpolation method. In section 3, a novel scheme is proposed to produce new optimized splines for interpolation regardless of the type of filtering. The performance of the proposed method is evaluated in section 4 by comparing the interpolation results of the proposed method on standard test images to those of well-known interpolation techniques. Section 5 concludes the paper.

2. PRELIMINARIES

In this paper, the following notation and definitions are used:

A discrete signal is shown by either $x[n] \triangleq x(nT)$ and is defined, $x_s(t) \triangleq x(t)s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$ where $s(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t-nT)$ is the comb function. Also, the period $T \triangleq 1$ is normalizes throughout the paper without any loss of generality.

Definition 1. For a discrete-time signal $x[n]$, S_x^m is a spline of order m if,

1. S_x^m would be a polynomial of the (at most) order m, in the interval $[n, n+1]$.
2. Interpolation property, $S_x^m(n) = x[n]$
3. $S_x^m \in C^{m-1}(-\infty, \infty)$

where $n \in \mathbb{Z}$.

If the goal is to discover a piecewise polynomial signal that is $m-1$ times differentiable with continuous derivatives, it is possible to calculate the integral of $c_x(t) = \sum_{n=-\infty}^{\infty} c_x[n]\delta(t-n)$, $m+1$ times, i.e,

$$S_x^m(t) = \left(u^{m+1} * c_x\right)(t) \quad (1)$$

where $u^1(t)$ is the unity step function and where $u^{k+1}(t) \triangleq \left(u^k * u^1\right)(t)$. Acording to the second condition, $S_x^m(t)s(t) = x_s(t)$, and substituting (1) in the last equation yeilds

$$x_s(t) = \left(u^{m+1} * c_x\right)(t)s(t) = \left((u^{m+1})_s * c_x\right)(t) \quad (2)$$

Hence, the above equation is satisfied, if the situation is defined $c_x(t) \triangleq \left(\left[(u^{m+1})_s\right]^{-1} * x_s\right)(t)$, where y^{-1} is the inverse of

y, i.e, $(y * y^{-1})(t) = \delta(t)$. Again, by substituting (1) it can be shown that

$$S_x^m(t) = \left(u^{m+1} * \left[(u^{m+1})_s\right]^{-1} * x_s\right)(t) \quad (3)$$

Definition 2. For every continuous-time signal $y(t)$ define,

$$L_y(t) \triangleq \left((y_s)^{-1} * y\right)(t) \quad (4)$$

If S_y^m is a spline of the order m, $L_{S_y^m}$ would be called a cardinal spline of order m. Hence, $L_{S_y^m}$ is independent of y and is only a function of m.

This new notation generates $S_x^m(t) = (L_{u^{m+1}} * x_s)(t)$. If $y[n]$ is an invertible discrete-time signal, S_y^m can be used to interpolate every discrete-time signal $x[n]$; i.e, it can be easily shown that $S_x^m(t) = \left(L_{S_y^m} * x_s\right)(t)$. Thus, L_m can be defined as the cardinal spline of the order m that for any continuous-time signal y

$$L_m \triangleq L_{S_y^m} \quad (5)$$

Assume β^m is an FIR filter and is a casual spline of order m that has non-zero values only for $0 \le t \le k \in \mathbb{N}$. It is desirable to find β^m as a basis for calculating other splines of order m from their samples instead of u^{m+1}. According to (1),

$$\beta^m(t) = u^{m+1} * \sum_{n=0}^{k} c_{\beta^m}[n]\delta(t - nT) \quad (6)$$

where $C_m(z)$ has been defined as the z-transform of the coefficients $c_{\beta^m}[n]$, i.e, $C_m(z) \triangleq \sum_{n=0}^{k} c_{\beta^m}[n]z^{-n}$. Thus, it can be shown that if β^m is an FIR, then $(z-1)^{m+1}$ divides $C_m(z^{-1})$. If one wished to minimize the length of β^m, it is possible use $C(z) \triangleq (z^{-1}-1)^{m+1}$, and $c_{\beta^m}[i]$ can be defined as $c_{\beta^m}[i] \triangleq (-1)^i\binom{m+1}{i}$. Moreover, if for every integer n we define $(x)_+^n = x^n u^1(x)$, then β^m can be rewritten as shown below

$$\beta^m(t) = \frac{1}{m!}\left[\sum_{n=0}^{m+1}(-1)^n\binom{m+1}{n}(t-n)_+^m\right] \quad (7)$$

3. THE PROPOSED OPTIMIZED B-SPLINE

In many applications, it is desirable that the interpolation filter be depicted as an ideal filter, and the second and third conditions of Definition 1 may not be important.

Assume that the goal is to design an optimized spline to interpolate $x[\cdot]$. Also assume that $\vec{b} \in \mathbb{R}^m$ be a vector such that $\sum_{k=1}^{m} b_k z^{-k}$ has no zeros on the unit circle.

Definition 3. $\beta_o^m\{x, \vec{b}\}$ is defined as follows,

1. $\beta_o^m\{x, \vec{b}\} \in C^0[0, m+1]$
2. $\beta_o^m\{x, \vec{b}\}[n] = \begin{cases} b_n & 1 \le n \le m \\ 0 & \text{o.w.} \end{cases}$
3. $e[\beta_o^m] \triangleq \int_{-\infty}^{\infty} |\mathcal{F}\{L_{\beta_o^m}\} - \frac{\mathcal{F}\{x\}}{\mathcal{F}\{x_s\}}|^2 df \to min$

where $\mathcal{F}\{y\}$ is the Fourier transform of y. Here, it has been assumed that β_m^o is known at the integers and thus, $(\beta_m^o)_s$ is also known. The fact that $\sum_{k=1}^{m} b_k z^{-k}$ has no roots on the unit circle implies that $(\beta_m^o)_s$ has a stable inverse and $L_{\beta_m^o}$ does exist.

Now, calculus variation may be used in order to evaluate the optimum β_m^o which minimizes the error $e[\beta_o^m]$. Considering $\gamma \in C^1[0, m+1]$ as a function which is zero at the integers, i.e, $\gamma(0) = \gamma(1) = ... = \gamma(m+1) = 0$. Variational derivation of e with respect to β_o^m with γ as a test function is equal to

$$\begin{aligned} \langle \delta e[\beta_o^m], \gamma \rangle &= \lim_{\varepsilon \to 0} \frac{e(\beta_o^m + \varepsilon\gamma) - e(\beta_o^m)}{\varepsilon} \\ &= 2\int_{-\infty}^{\infty} \gamma(-t)\Re\Bigg\{\mathcal{F}^{-1}\Bigg\{\left[\frac{\mathcal{F}\{x_s\}}{\mathcal{F}\{(\beta_o^m)_s\}}\right]^* \\ &\quad \left[\mathcal{F}\{L_{\beta_o^m}\}\mathcal{F}\{x_s\} - \mathcal{F}\{x\}\right]\Bigg\}\Bigg\}dt \end{aligned} \quad (8)$$

Note that according to (4), $\mathcal{F}\{L_{\beta_o^m}\} = \frac{\mathcal{F}\{\beta_o^m\}}{\mathcal{F}\{(\beta_o^m)_s\}}$. In order to minimize $e[\beta_o^m]$, $\langle \delta e[\beta_o^m], \gamma \rangle$ should be zero for all γ, which implies that the second term inside the integral should be zero for $t \in (0, m+1)$; hence,

$$(x_s * \overline{x_s}) * [(\beta_o^m)_s^{-1} * (\overline{\beta_o^m})_s^{-1}] * \beta_o^m = [(\overline{\beta_o^m})_s^{-1} * \overline{x_s}] * x\Big|_{t \in (0, m+1)} \quad (9)$$

where $\overline{y}(t) \triangleq y(-t)$. Thus, it is proven that the optimized B-spline which could give the best estimation of x, should satisfy (9). By defining $a(t) \triangleq (x_s * \overline{x_s}) * [(\beta_o^m)_s^{-1} * (\overline{\beta_o^m})_s^{-1}]$ and $c(t) \triangleq [(\overline{\beta_o^m})_s^{-1} * \overline{x_s}]$, (9) can be written as $a * \beta_o^m = c|_{t \in (0, m+1)}$. In order to derive β_o^m from (9), for $n \in \mathbb{Z}$, two sequences of functions defined as $B_n(t) \triangleq \beta(t+n)[u^1(t) - u^1(1-t)]$ and $C_n(t) \triangleq c(t+n)[u^1(t) - u^1(1-t)]$. Now, (9) can be written in a matrix form as follows

$$\begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_m \end{bmatrix} = \begin{bmatrix} a[0] & a[-1] & \dots & a[-m] \\ a[1] & a[0] & \dots & a[-m+1] \\ \vdots & \vdots & \ddots & \vdots \\ a[m] & a[m-1] & \dots & a[0] \end{bmatrix}^{-1} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_m \end{bmatrix} \quad (10)$$

According to (10), $\{B_n\}_{n=1}^{\infty}$ is derived and thus the optimized B-spline is evaluated as $\beta_o^m(t) = \sum_{n=0}^{m} B_n(t-n)$.

Now in order to design β_o^m such that the frequency response of $L_{\beta_o^m}$ would be the best estimation for the filter that has the interpolation property, and h is the impulse response,

$$h[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{o.w.} \end{cases} \Rightarrow h_s(t) = \delta(t) \quad (11)$$

If a desired impulse response h is substituted for x in (9), then for the optimized $\beta_o^m\{h, \vec{b}\}$, the following expression is minimized:

$$e[\beta_o^m\{h, \vec{b}\}] = \int_{-\infty}^{\infty} |\mathcal{F}\{L_{\beta_o^m}\} - \mathcal{F}\{h\}|^2 df \quad (12)$$

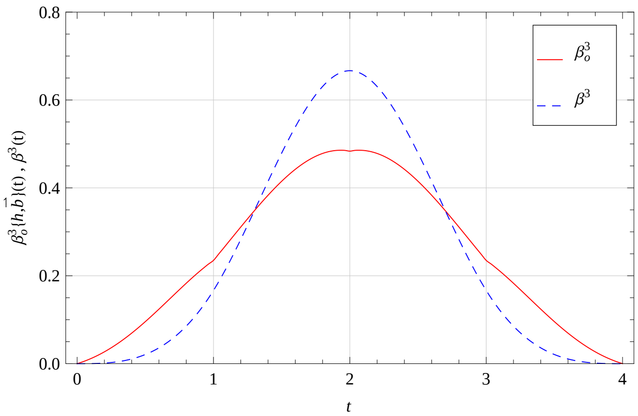

Fig. 1. The optimized spline versus B-spline, of order three. $\beta_o^3\{h, \vec{b}\}$ is the optimized basis spline built for estimating the ideal lowpass filter $h(t) = \frac{sin(\pi t)}{\pi t}$ with $\vec{b} = (0.235, 0.484, 0.235)$

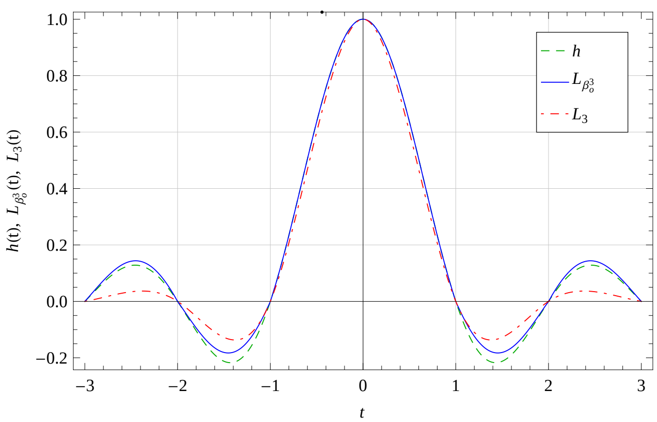

Fig. 2. The comparison of the proposed method and the cubic spline for ideal lowpass filter.

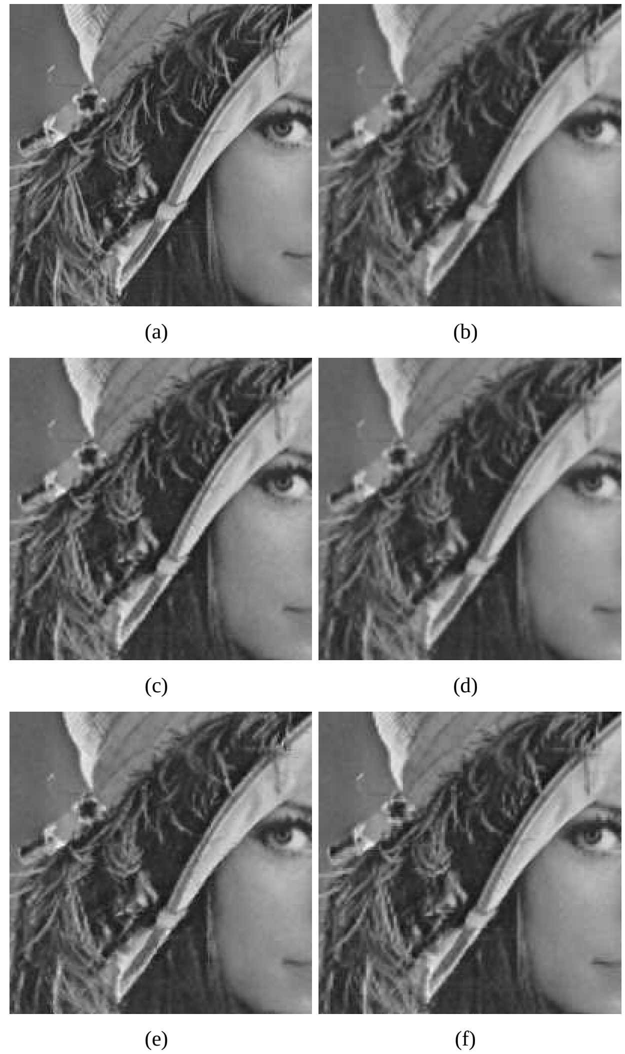

Fig. 3. Comparison of different methods for the Lena image: (a) The original image, (b) bilinear interpolation, (c) bicubic Interpolation. (d) WZP Cycle-Spinning [12], (e) SAI [19], and (f) the proposed method.

Thus, β_o^m is the most proper basis for estimating the interpolation by the ideal filter h. Since $h_s(t) = \delta(t)$, producing

$$[(\beta_o^m)_s^{-1} * (\overline{\beta_o^m})_s^{-1}] * \beta_o^m = (\overline{\beta_o^m})_s^{-1} * h\Big|_{t\in(0,m+1)} \quad (13)$$

4. SIMULATION RESULTS

The performance of the proposed method for an ideal lowpass filter has been compared to the B-spline and the results are depicted in Figs. 1 and 2. Fig. 1 shows the comparision of the optimized basis spline built for estimating an ideal lowpass filter and the cubic B-spline. Fig. 2 shows $L_{\beta_o^m}$ as compared to L_3. The optimized spline is superior to the B-spline method. The SNR values of these methods are 20.39dB and 13.15dB for the proposed method and the B-spline method, respectively, for $m = 3$.

To consider practical applications, the method was tested on several standard monochrome images. These images are down-sampled to provide the low solution images for interpolation. In image applications, splines can be used for zooming and enlargements. For comparison, three other image interpolation methods are also simulated: 1-bicubic interpolation, 2-wavelet-domain zero padding cycle-spinning [12] and 3-soft-decision estimation technique for adaptive image interpolation [19]. Table 4 shows the Peak Signal-to-Noise Ratio (PSNR) performance of these three methods when applied to the seven well-known test images. In all cases, the proposed optimized spline interpolation algorithm performed best among all methods. For high frequency content images, such as Barbara and Baboon, the proposed algorithm outperforms other methods by 1dB.

Since PSNR is an average quality measure, the spatial locations where the proposed algorithm produces significantly smaller interpolation errors than the other competing methods are plotted in Fig. 3. The differences are more noticeable around the edge of the hat. The result of the present study com-

Table 1. PSNR (dB) Results of the Reconstructed Images by Various Methods (Image Enlargement from 256×256 to 512×512)

Images	Bicubic [6]	WZP–CS [12]	SAI [19]	Opt.Spline
Lena	30.13	30.05	30.88	**32.29**
Baboon	21.34	21.70	22.09	**22.50**
Barbara	23.32	23.88	23.71	**25.10**
Peppers	28.61	28.60	28.91	**30.64**
Couple	26.73	26.86	26.96	**27.91**
Bout	26.93	27.07	27.63	**28.50**
Girl	29.97	30.20	29.94	**30.90**

pare favorably both subjectively and objectively. In addition, a wavelet scheme based on cycle-spinning interpolation has been included to provide a comparison with a powerful method operating in the wavelet domain.

5. CONCLUSION

This paper has introduced a method for optimizing a compact support interpolating spline for approximating a given filter in the least square sense. In particular,it demonstrated a newly proposed method for approximating the ideal lowpass filter. The interpolation results obtained by this method are better than those obtained by the conventional solutions, such as B-splines. Simulation results show about 1dB improvement in most of the cases. In the future, we plan to focus on the application of these optimized splines for non-uniform sampling for 1-D and 2-D signals.

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