A New Achievable Rate Region for the Gaussian Two-Way Relay Channel via Hybrid Broadcast Protocol

Shahab Ghasemi-Goojani, *Member, IEEE*, and Hamid Behroozi, *Member, IEEE*

***Abstract*—In this letter, we study the Gaussian two-way relay channel (GTWRC) with a direct link in which two nodes want to exchange information with each other with the help of a relay node in the presence of a direct link between two nodes. Specially, we focus on a protocol with four phases, which is called the hybrid broadcast (HBC) protocol. In the HBC protocol, sequential transmissions from both users are followed by a transmission from the relay. Using nested lattice codes, we obtain a new achievable rate region for this protocol. In fact, utilizing a four-stage lattice partition chain, we introduce an intermediate lattice in our lattice-based coding scheme, and by using it, we split the codebook of each node judiciously into two parts. This enables us to decode a linear combination of lattice codewords partially at the relay. Using our proposed strategy, we show that the achievable rate region is superior to the obtained achievable rate region by Kim, Mitran, and Tarokh.**

***Index Terms*—Two-way relay channel, hybrid broadcast protocol, rate region, four-stage lattice partition chain, nested lattice codes.**

I. Introduction

The two-way relay channel (TWRC) in which two nodes wish to exchange messages with the help of a relay node is a practical channel model (such as 4G wireless standards) for wireless communication systems. In this letter, we consider the GTWRC with a direct link between two users which operates under the half duplex mode in which each node can either transmit or receive but not both at the same time.

A large number of coding schemes has been proposed for the half-duplex GTWRC, see, e.g., [1] and [2]. The basic protocol for the TWRC, which consists of two phases is multiple access and broadcast (MABC) protocol (see phases 3 and 4 of Fig. 1). In the literature, several network coding schemes for this setup are investigated (see, e.g., [3]–[5]). For the broadcast phase of the GTWRC, the capacity region in terms of the maximal probability of error is completely characterized in [3]. An approach based on the physical layer network coding is considered in [4]. A strategy based on symbol-wise network coding is investigated in [5]. In [6] and [7], it is shown that lattice codes can achieve the capacity region of the GTWRC (without the direct link) within 0.5 bit for symmetric and asymmetric case, respectively. At [8] using Partial Decode-and-Forward some rate-regions for the GTWRC in both full- and half-duplex modes is obtained.

Manuscript received March 31, 2014; revised August 22, 2014; accepted August 25, 2014. Date of publication September 4, 2014; date of current version November 7, 2014. The associate editor coordinating the review of this paper and approving it for publication was T. J. Oechtering.

The authors are with the Department of Electrical Engineering, Sharif University of Technology, Tehran 11365-11155, Iran (e-mail: ghasemeishahab@gmail.com; behroozi@sharif.edu).

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Digital Object Identifier 10.1109/LCOMM.2014.2354362

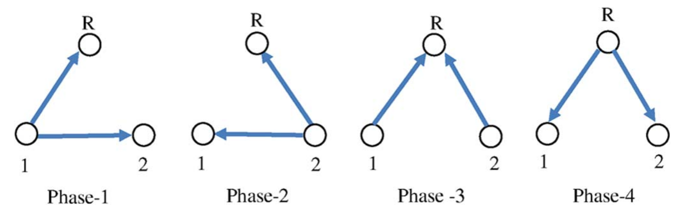

Fig. 1. The HBC protocol for the Gaussian two-way relay channel.

Two well-known decode-and-forward protocols with a direct link between users which are referred to as the time division broadcast (TDBC) and the hybrid broadcast (HBC), respectively, are considered in [9] (see Fig. 1 for the HBC protocol. For the TDBC protocol, we use only phases 1, 2 and 4). A six-phase protocol is considered at [10] and it is shown that this protocol can achieve rate-regions larger than that of the HBC protocol.

In this letter, instead of increasing transmission phases (which leads to a protocol with higher complexity) as in [10], we obtain two rate-regions for the HBC protocol using nested lattice codes: The first is based on a direct extension of the proposed scheme at [7] while at our proposed scheme, by introducing an intermediate lattice, we split the codebook of each node to decode linear combination of codewords partially at the relay. Using numerical simulations, we show that our scheme is completely better than that of a direct applying of the proposed scheme at [7].

In our scheme, using the intermediate lattice, we divide any lattice codeword into two parts. At phases 1 and 2, each node sends its message and recovers the first part of messages. In the 3rd phase, each node sends the corresponding lattice codeword and using the idea of computation coding [11] and the side information obtained at phases 1 and 2, the relay node recovers a linear combination of the second parts of lattice codewords. As the final phase, the relay first constructs a fine linear combination and sends it over channel. Since each node has its own message, it decodes the desired message. We show that our achievable rate-region is larger than of [9] and however only uses four phases achieves a larger rate-region than the six-phase protocol in [10] under some constraints over rates. Note that this performance enhancement is not possible with the proposed scheme in [9]. Finally, using our scheme, one can improve the obtained rate-region for the GTWRC in [8] in both full- and half-duplex modes.

II. Channel Model

This letter studies the GTWRC under the half duplex mode, i.e., each node can only either listen or transmit at the same time. In the GTWRC, node 1 and 2 intend to exchange messages $W_1 \in \{1, 2, \ldots, 2^{nR_1}\}$ and $W_2 \in \{1, 2, \ldots, 2^{nR_2}\}$ with the assistance of a relay (represented by node r). The relative time duration of $m-th$ phase is denoted by t_m. We consider the GTWRC which operates in four phases, as depicted in

Fig. 1. We refer to this protocol as the HBC protocol which is described by the following equations:

$$\text{Phase } 1: \boldsymbol{Y}_2^{(1)} = g_3\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_2^{(1)}\boldsymbol{Y}_r^{(1)} = g_1\boldsymbol{X}_1^{(1)} + \boldsymbol{Z}_r^{(1)},$$
$$\text{Phase } 2: \boldsymbol{Y}_1^{(2)} = g_3\boldsymbol{X}_2^{(2)} + \boldsymbol{Z}_1^{(2)}\boldsymbol{Y}_r^{(2)} = g_2\boldsymbol{X}_2^{(2)} + \boldsymbol{Z}_r^{(2)},$$
$$\text{Phase } 3: \boldsymbol{Y}_r^{(3)} = g_1\boldsymbol{X}_1^{(3)} + g_2\boldsymbol{X}_2^{(3)} + \boldsymbol{Z}_r^{(3)},$$
$$\text{Phase } 4: \boldsymbol{Y}_1^{(4)} = g_1\boldsymbol{X}_r^{(4)} + \boldsymbol{Z}_1^{(4)}\boldsymbol{Y}_2^{(4)} = g_2\boldsymbol{X}_r^{(4)} + \boldsymbol{Z}_2^{(4)},$$

where $g_i(i=1,2)$ denotes the channel gain between node i and the relay node, and g_3 denotes the channel gain between nodes 1 and 2. $\boldsymbol{X}_i^{(m)}$ and $\boldsymbol{Y}_i^{(m)}$ are transmit and received signals of node $i(i=1,2,r)$ in phase m, respectively, and $\boldsymbol{Z}_i^{(m)}$ is an i.i.d. Gaussian noise with zero mean and variance N_i in phase m. We assume that node i has the transmit power P_i for each phase.

III. Achievable Rate Region

In this section, using a lattice-based coding scheme, we obtain two achievable rate-regions for the HBC protocol. For the preliminaries of lattice codes, due to the limited space, we refer the interested reader to [11] and [12]. The first scheme is based on the direct extension of the scheme in [7] for the full duplex GTWRC without a direct link while in the proposed scheme, we use an intermediate lattice in order to split the codewords.

A. Rate-Region I: A Direct Extension of the Scheme in [7]

In this extension, we use rate-splitting at both nodes, i.e., message $W_i(i=1,2)$ is represented by message W_{ai} at rate R_{ai} and message W_{bi} at rate R_{bi}. Thus, $R_1=R_{a1}+R_{b1}$ and $R_2=R_{a2}+R_{b2}$. In phase 1, node 1 transmits its codeword for message W_{a1}, i.e., $\boldsymbol{X}_{a1}(W_{a1})$ at rate R_{a1} using random Gaussian codes. Similarly, node 2 transmits its codeword for message W_{a2}, i.e., $\boldsymbol{X}_{a2}(W_{a2})$ at rate R_{a2} in phase 2 using random Gaussian codes. At the end of phase $i(i=1,2)$, node $j(j\in\{1,2\}, j\neq i)$ can estimate the desired message correctly if

$$R_{ai} \leq t_i C\left(\frac{g_3^2 P_i}{N_j}\right) \tag{1}$$

where $C(x) = \frac{1}{2}\log(1+x)$. In order to communicate messages W_{b1} and W_{b2} in phases 3 and 4, we operate exactly the same as the proposed scheme in [7]. In phase 3, using nested lattice codes, nodes 1 and 2 map messages $W_{b1}$and W_{b2} to lattice points and transmit at R_{b1} and R_{b2}, respectively, and the relay decodes the modulo lattice summation. In phase 4, the relay broadcasts the modulo summation. From [7], we can see that the following rate-region is achievable:

$$R_{bi} \leq \min\left(t_3\frac{1}{2}\log\left(\frac{g_i^2P_i}{g_1^2P_1+g_2^2P_2}+\frac{g_i^2P_i}{N_r}\right), t_4C\left(\frac{g_j^2P_r}{N_j}\right)\right), \tag{2}$$

with $i,j\in\{1,2\}$ and $i\neq j$. Since $R_i = R_{ai}+R_{bi}$, using (1) and (2), the following rate-region is achievable

$$R_i \leq t_iC\left(\frac{g_3^2P_i}{N_j}\right) + \min\left(t_3\frac{1}{2}\log\left(\frac{g_i^2P_i}{g_1^2P_1+g_2^2P_2}+\frac{g_i^2P_i}{N_r}\right), t_4C\left(\frac{g_j^2P_r}{N_j}\right)\right), \tag{3}$$

where $i,j\in\{1,2\}$ and $i\neq j$. Note that this scheme completely ignores the relay node at phases 1 and 2.

B. Rate-Region II: Our Proposed Scheme

The general idea behind our transmission scheme is that each transmitter divides the message into two parts using lattice codes. In phases 1 and 2, each node sends its message and then at the end of each phase, it decodes the first part of message. The relay node also similarly decode the first part of message at phases 1 and 2. In the third phase, users 1 and 2 send the entire messages to the relay. Using the obtained side-information at phases 1 and 2, the relay is able decode a linear combination of the second part of messages. In phase 4, the relay node first constructs a fine linear combination of messages and then sends this linear combination using random coding. Since each node knows its own message, it can decode the desired message easily. In the following, we explain our scheme in more details.

Without loss of generality, we assume that $g_1^2P_1\geq g_2^2P_2$. Then, from [13], we choose four nested lattices Λ_1, Λ_2, Λ_b and Λ_c such that $\Lambda_1\subseteq\Lambda_2\subseteq\Lambda_b\subseteq\Lambda_c$ where Λ_c is Poltyrev-good while lattices Λ_b, Λ_1 and Λ_2 are simultaneously Rogers-good and Poltyrev-good. Lattices Λ_1 and Λ_2 have the following second moments: $\sigma^2(\Lambda_i)=g_i^2P_i, (i=1,2)$. By lattices Λ_1 and Λ_2, we satisfy the channel power constraints at nodes 1 and 2, respectively, and using lattice Λ_c, we generate codewords. On the other hand, to divide the lattice points at each node, we use intermediate lattice Λ_b. Based on the four-stage lattice partition chain $\Lambda_1\subseteq\Lambda_2\subseteq\Lambda_b\subseteq\Lambda_c$, each node generates a codebook as:

$$\mathcal{C}_i = \{\Lambda_c\cap\mathcal{V}_i\},\quad i=1,2 \tag{4}$$

where $\mathcal{V}_i$ denotes the Voronoi region of lattice Λ_i. Now, using the intermediate lattice Λ_b, we divide $\mathcal{C}_i$ as follows:

$$\mathcal{C}_a = \{\mathcal{V}_b\cap\Lambda_c\}, \mathcal{C}_{bi} = \{\mathcal{V}_i\cap\Lambda_b\}. \tag{5}$$

Considering the *nearest neighbor quantizer* $\mathcal{Q}_\Lambda$ that maps any point $\boldsymbol{x}\in\mathbb{R}^n$ to the nearest lattice point, i.e., $\mathcal{Q}_\Lambda(\boldsymbol{x}) = \arg\min_{\boldsymbol{l}\in\Lambda}\|\boldsymbol{x}-\boldsymbol{l}\|$, and the *modulo-*$\Lambda$*operation* with respect to lattice Λ that returns the quantization error, i.e., $[\boldsymbol{x} \bmod \Lambda] = \boldsymbol{x}-\mathcal{Q}(\boldsymbol{x})$, we can write any lattice point $\boldsymbol{V}_i\in\mathcal{C}_i$ as below:

$$\boldsymbol{V}_i = [\boldsymbol{V}_{ai}+\boldsymbol{V}_{bi}] \bmod \Lambda_i, \tag{6}$$

where

$$\boldsymbol{V}_{ai} = \boldsymbol{V}_i \bmod \Lambda_b, \boldsymbol{V}_{bi} = [\boldsymbol{V}_i - \boldsymbol{V}_{ai}] \bmod \Lambda_i. \tag{7}$$

We denote the associated rate of lattice points $\boldsymbol{V}_a$ and $\boldsymbol{V}_b$ by $R_{ai}^{(m)}$ and $R_{bi}^{(m)}$ for user i at phase m. The overall rate for user i which is the rate of any lattice point $\boldsymbol{V}_i\in\mathcal{C}_i$ is denoted by R_i. Before presenting our proposed scheme, note that node i maps message W_i to lattice point $\boldsymbol{V}_i\in\mathcal{C}_i$ using a one-to-one mapping. Thus, in the following, we only focus on recovering the lattice point $\boldsymbol{V}_i$. On the other hand, by calculating the optimum phase durations, t_1, t_2, t_3 and t_4, we can determine the codeword length at each phase as $n_1=\frac{t_1}{T_s}$, $n_2=\frac{t_2}{T_s}$, $n_3=\frac{t_3}{T_s}$, and $n_4=\frac{t_4}{T_s}$, where T_s is the sampling interval. Accordingly, transmitted sequences as well as lattices in phase m have the block length n_m.

1) Phases 1 and 2: In phase i $(i=1,2)$, node i sends the lattice points $\boldsymbol{V}_{ai}$ and $\boldsymbol{V}_{bi}$ using superposition coding as the following

$$\boldsymbol{X}_i^{(i)} = \boldsymbol{X}_{ai}(\boldsymbol{V}_{ai}) + \boldsymbol{X}_{bi}(\boldsymbol{V}_{bi}),$$

where $X_{ai}\sim\mathcal{N}(0,\alpha P_i)$ and $X_{bi}\sim\mathcal{N}(0,\bar{\alpha}P_i)$ are independent and $\alpha\in[0,1]$. The relay node, using the joint-typicality argument, can correctly estimate $\boldsymbol{V}_{ai}$ if [14]

$$R_{ai}^{(i)} \leq C\left(\frac{\alpha g_i^2 P_i}{\bar{\alpha} g_i^2 P_i + N_r}\right). \qquad i = 1, 2. \tag{8}$$

Note that at the relay node, we only decode $\boldsymbol{V}_{ai}$. Thus, the preceding constraints are over the related rates with $\boldsymbol{V}_{ai}$. On the other hand, at the end of phase 1 (2), node 2 (1) does not recover $\boldsymbol{V}_1(\boldsymbol{V}_2)$. However, at the end of phase 4, the received signal $\boldsymbol{Y}_2^{(1)}(\boldsymbol{Y}_1^{(2)})$ is used to decode the lattice point $\boldsymbol{V}_1\boldsymbol{V}_2)$.

2) Phase 3: In this phase, node i sends $\boldsymbol{V}_i$ to the relay node. At the relay node, instead of decoding the information sent by the other nodes separately, it decodes a linear combination of the lattice points $\boldsymbol{V}_{b1}$ and $\boldsymbol{V}_{b2}$. In more details, node i sends the following sequence over the channel

$$\boldsymbol{X}_i^{(3)} = \frac{1}{g_i}[\boldsymbol{V}_i - \boldsymbol{D}_i] \bmod \Lambda_i, \; i = 1, 2, \tag{9}$$

where $\boldsymbol{D}_i \sim \text{Unif}(\mathcal{V}_i)$ and is known at both nodes and at the relay. Based on the Crypto lemma [12], we know that the power constraint at each node is satisfied. The relay by receiving $\boldsymbol{Y}_r^{(3)}$ tries to decode a linear combination of $\boldsymbol{V}_{b1}$ and $\boldsymbol{V}_{b2}$. To do this, since the relay knows $\boldsymbol{V}_{a1}$ and $\boldsymbol{V}_{a2}$ from the previous phases, it first cancels their effects and performs the following operations:

$$\begin{aligned}
\boldsymbol{Y}_{dr}^{(3)} &= \left[\boldsymbol{Y}_r^{(3)} - \boldsymbol{V}_{a1} - \boldsymbol{V}_{a2} + \boldsymbol{D}_1 + \boldsymbol{D}_2\right] \bmod \Lambda_1 \\
&= \Big[g_1\boldsymbol{X}_1^{(3)} + g_2\boldsymbol{X}_2^{(3)} + \boldsymbol{Z}^{(3)} - \boldsymbol{V}_{a1} - \boldsymbol{V}_{a2} \\
&\quad + \boldsymbol{D}_1 + \boldsymbol{D}_2] \bmod \Lambda_1 \\
&= [(\boldsymbol{V}_1 - \boldsymbol{V}_{a1}) + (\boldsymbol{V}_2 - \boldsymbol{V}_{a2}) \\
&\quad - \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{D}_2) + \boldsymbol{Z}^{(3)}\Big] \bmod \Lambda_1 \\
&= [\boldsymbol{V}_{b1} + \boldsymbol{V}_{b2} + \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{V}_{a2}) \\
&\quad - \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{D}_2) + \boldsymbol{Z}^{(3)}\Big] \bmod \Lambda_1,
\end{aligned} \tag{10}$$

where $\mathcal{Q}_{\Lambda_2}(.)$ shows the nearest neighbor quantizer w.r.t. lattice Λ_2 and (10) is based on the distributive law for the modulo operation [11]and the fact that $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_b$. Since

$$\begin{aligned}
\boldsymbol{V}_{b-sum} \triangleq [&\boldsymbol{V}_{b1} + \boldsymbol{V}_{b2} + \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{V}_{a2}) \\
&- \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{D}_2)] \bmod \Lambda_1 \in \Lambda_b
\end{aligned} \tag{11}$$

we can correctly decode $\boldsymbol{V}_{b-sum}$ if and only if lattice Λ_b is Poltyrev-good, i.e., [15]

$$\frac{\text{Vol}\left(\mathcal{V}_b\right)^{\frac{2}{n}}}{2\pi e \text{Var}\left(\boldsymbol{Z}^{(3)}\right) \geq 1,} \tag{12}$$

where $\text{Vol}(\mathcal{V}_b)$ denotes the Voronoi region volume of lattice Λ_b. Now, we calculate the rate $R_{bi}^{(3)}$ as the following

$$\begin{aligned}
R_{bi}^{(3)} &= \frac{1}{n}\log\left(\frac{\text{Vol}(\mathcal{V}_i)}{\text{Vol}(\mathcal{V}_b)}\right) = \frac{1}{2}\log\left(\frac{\sigma^2(\Lambda_i)}{G(\Lambda_i)\text{Vol}(\mathcal{V}_b)^{\frac{2}{n}}}\right) && (13) \\
&\leq \frac{1}{2}\log\left(\frac{\sigma^2(\Lambda_i)}{G(\Lambda_i)2\pi e \text{Var}\left(\boldsymbol{Z}^{(3)}\right)}\right) && (14) \\
&= \left[\frac{1}{2}\log\left(\frac{g_i^2 P_i}{N_r}\right)\right]^+, && (15)
\end{aligned}$$

where (13) is based on definition of the normalized second moment and (14) follows from (12). (15) is based on the fact that lattice Λ_i is Rogers-good, i.e., $G(\Lambda_i) \to \frac{1}{2\pi e}$.

The advantage of our proposed scheme is enabling us to recover $\boldsymbol{V}_{b-sum}$ in another way. In fact, since we have applied superposition coding at phases 1 and 2, by first decoding $\boldsymbol{V}_{b1}$ and $\boldsymbol{V}_{b2}$ in this phase and then using $\boldsymbol{V}_{a1}$ and $\boldsymbol{V}_{a2}$ which are decoded at the previous phases and using dithers which are known at the relay node, we can easily recover $\boldsymbol{V}_{b-sum}$. Therefore, the required rate-region for constructing this signal is:

$$R_{bi}^{(i)} \leq C\left(\frac{\bar{\alpha} g_i^2 P_i}{N_r}\right). \qquad \forall i = 1, 2. \tag{16}$$

Now, using time sharing between (15) and (16), we get the following rate-region for estimating $\boldsymbol{V}_{b-sum}$

$$R_{bi} \leq t_i C\left(\frac{\bar{\alpha} g_i^2 P_i}{N_r}\right) + t_3\left[\frac{1}{2}\log\left(\frac{g_i^2 P_i}{N_r}\right)\right]^+. \tag{17}$$

The relay node knows $\boldsymbol{V}_{a1}$ and $\boldsymbol{V}_{a2}$ from phases 1 and 2. Thus, in order to send its information at phase 4, first constructs the following signal:

$$\begin{aligned}
\boldsymbol{V}_{Sum} &\triangleq [\boldsymbol{V}_{b-sum} + \boldsymbol{V}_{a1} + \boldsymbol{V}_{a2}] \bmod \Lambda_1, \\
&= [[\boldsymbol{V}_{a1} + \boldsymbol{V}_{b1}] + [\boldsymbol{V}_{a2} + \boldsymbol{V}_{b2}] + \\
&\quad \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{V}_{a2}) - \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{D}_2)] \bmod \Lambda_1 \\
&= [\boldsymbol{V}_1 + \boldsymbol{V}_2 + \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_{a2} + \boldsymbol{V}_{b2}) + \\
&\quad \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{V}_{a2}) - \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{D}_2)] \bmod \Lambda_1
\end{aligned} \tag{18}$$

where (18) follows from the distributive law for the modulo.

3) Phase 4: In this phase, the relay node using random coding starts to broadcast its understanding from the previously received signals, $\boldsymbol{V}_{Sum}$, to the users. Based on our first assumption on power constraints, $R_1 \geq R_2$. Thus, by the Crypto lemma, the cardinality of $\boldsymbol{V}_{Sum}$ is $2^{n_3 R_1^{(3)}}$.

We choose the Gaussian distribution $P(x_r)$ with mean zero and variance P_r and generate $2^{n_4 R_1^{(4)}}$ codewords $\boldsymbol{X}_r^{(4)}$ using this distribution. The relay maps $\boldsymbol{V}_{Sum}$ to $\boldsymbol{X}_r^{(4)}(\boldsymbol{V}_{Sum})$ and then sends it over the channel. Note that for having a one-to-one mapping between lattice codewords $\boldsymbol{V}_{Sum}$ and $\boldsymbol{X}_r^{(4)}(\boldsymbol{V}_{Sum})$, we must have $n_3 R_1^{(3)} = n_4 R_1^{(4)}$. Node i by receiving the sequences $\boldsymbol{Y}_i^{(j)}$ and $\boldsymbol{Y}_i^{(4)}$, where $i, j \in \{1, 2\}$ and $i \neq j$, at phases j and 4, estimates the message of node j, $\boldsymbol{V}_j$, as $\widehat{\boldsymbol{V}}_j$ if two unique codewords $\boldsymbol{X}_r^{(4)}(\widehat{\boldsymbol{V}}_{Sum}) \in \mathcal{C}_i^{(4)}$and $\boldsymbol{X}_j^{(j)}(\widehat{\boldsymbol{V}}_j)$ exist such that $(\boldsymbol{X}_r^{(4)}(\widehat{\boldsymbol{V}}_{Sum}), \boldsymbol{X}_j^{(j)}(\widehat{\boldsymbol{V}}_j), \boldsymbol{Y}_i^{(j)}, \boldsymbol{Y}_i^{(4)})$ are jointly typical, where

$$\begin{aligned}
\mathcal{C}_1^{(4} = \Big\{&\boldsymbol{X}_r^{(4)}(\widehat{\boldsymbol{V}}_{Sum}) : \widehat{\boldsymbol{V}}_{Sum} = [\boldsymbol{V}_1 + \boldsymbol{v}_2 + \mathcal{Q}_{\Lambda_2}(\boldsymbol{v}_{a2} + \boldsymbol{v}_{b2}) \\
&+ \mathcal{Q}_{\Lambda_2}(\boldsymbol{v}_2 - \boldsymbol{v}_{a2}) - \mathcal{Q}_{\Lambda_2}(\boldsymbol{v}_2 - \boldsymbol{D}_2)] \bmod \Lambda_1, \boldsymbol{v}_2 \in \mathcal{C}_2\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}_2^{(4)} = \Big\{&\boldsymbol{X}_r^{(4)}(\widehat{\boldsymbol{V}}_{Sum}) : \widehat{\boldsymbol{V}}_{Sum} = [\boldsymbol{v}_1 + \boldsymbol{V}_2 + \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_{a2} + \boldsymbol{V}_{b2}) \\
&+ \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{V}_{a2}) - \mathcal{Q}_{\Lambda_2}(\boldsymbol{V}_2 - \boldsymbol{D}_2)] \bmod \Lambda_1, \boldsymbol{v}_1 \in \mathcal{C}_1\Big\}.
\end{aligned}$$

Note that the cardinality of codebook $\mathcal{C}_i^{(4)}$ is $2^{n_4 R_i^{(4)}}$. From the argument of random coding/jointly typical decoding [14], node i can recover $\widehat{\boldsymbol{V}}_j$ if

$$R_i \leq t_i C\left(\frac{g_3^2 P_i}{N_j}\right) + t_4 C\left(\frac{g_j^2 P_r}{N_j}\right). \tag{19}$$

For calculating the preceding rate, note that the codewords $\boldsymbol{X}_j^{(j)}$and $\boldsymbol{X}_r^{(4)}$ are independent.

IV. Achievable Rate-Region for the HBC Protocol

By applying the Fourier–Motzkin elimination method, from (8), (17), and (19), we get the following rate for R_i:

$$R_i \leq \min\left\{ t_i C\left(\frac{g_i^2 P_i}{N_r}\right) + t_3\left[\frac{1}{2}\log\left(\frac{g_i^2 P_i}{N_r}\right)\right]^+, \right.$$
$$\left. t_i C\left(\frac{g_3^2 P_i}{N_j}\right) + t_4 C\left(\frac{g_j^2 P_r}{N_j}\right)\right\}, \quad (20)$$

where $i, j \in \{1, 2\}$ and $i \neq j$. For comparison, in addition to the achievable rate-region 1 in (3), we have the following achievable rate-region for the HBC protocol in [9]:

$$R_i \leq \min\left\{ t_i C\left(\frac{g_i^2 P_i}{N_r}\right) + t_3 C\left(\frac{g_i^2 P_i}{N_r}\right), \right.$$
$$\left. t_i C\left(\frac{g_3^2 P_i}{N_j}\right) + t_4 C\left(\frac{g_j^2 P_r}{N_j}\right)\right\},$$
$$R_1 + R_2 \leq t_1 C\left(\frac{g_1^2 P_1}{N_r}\right) + t_2 C\left(\frac{g_2^2 P_2}{N_r}\right)$$
$$+ t_3 C\left(\frac{g_1^2 P_1 + g_2^2 P_2}{N_r}\right), \quad (21)$$

with $i, j \in \{1, 2\}$ and $i \neq j$. By comparing the achievable rate-region given in (21) and that of our proposed scheme, given in (20), we can see that the constraints over R_1 and R_2 at our scheme are similar to those of (21) but our scheme by removing the constraint on the sum-rate achieves a rate-region larger than (21). To evaluate the gap between our achievable rate region, given in (20) and rate-region 1, given in (3), it is sufficient to assume that $g_3 \ll 1$. Then, by assuming $t_4 C\left(\frac{g_2^2 P_r}{N_2}\right) > t_1 C\left(\frac{g_1^2 P_1}{N_r}\right) + t_3\left[\frac{1}{2}\log\left(g_1^2 P_1 N_r\right)\right]^+$ and $t_4 C\left(\frac{g_1^2 P_r}{N_1}\right) > t_2 C\left(\frac{g_2^2 P_2}{N_r}\right) + t_3\left[\frac{1}{2}\log\left(\frac{g_2^2 P_2}{N_r}\right)\right]^+$ we can see that the gap for users 1 and 2 is $t_1 C\left(\frac{g_1^2 P_1}{N_r}\right)$ and $t_2 C\left(\frac{g_2^2 P_2}{N_r}\right)$, respectively, which can be significant for typical SNRs and it can grow to infinity as $\mathrm{SNR} \to \infty$.

In Fig. 2, we compare the achievable rate-region of scheme I and scheme 2 with that of the proposed scheme in [9] as well as the proposed scheme in [10]. We observe that our proposed scheme achieves a larger rate-region than scheme I and the scheme proposed in [9]. Although the HBC protocol only uses four phases, we observe that under some conditions over rates, our proposed scheme can achieve a larger rate-region than the proposed scheme in [10] (which uses a six-phase protocol).

V. Conclusion

In this letter, we considered the GTWRC with direct link which operates at four phases. This setup is called the HBC protocol that is introduced in [9]. Using nested lattices, we proposed a new achievable rate-region, which is a larger rate-region compared with the rate-region given in [9]. Although the HBC protocol uses only four phases, we observe that under some conditions, it is able to achieve a larger rate-region compared with the 6-phase protocol proposed in [10].

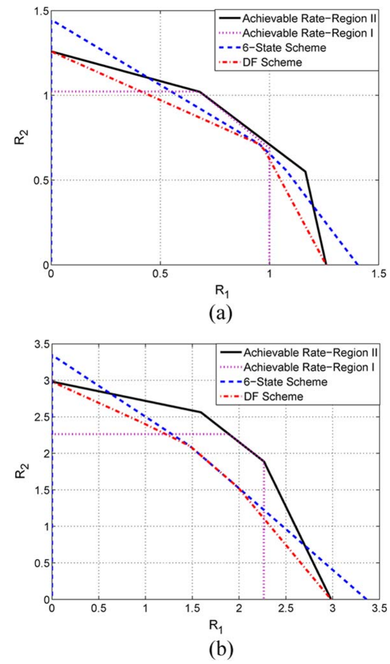

Fig. 2. Comparison of bi-directional regions with $P_1 = P_2 = P_r = 5$ dB. (a) $\frac{g_1^2}{N_r} = \frac{g_1^2}{N_1} = 5$ dB, $\frac{g_2^2}{N_r} = \frac{g_2^2}{N_2} = 10$ dB and $\frac{g_3^2}{N_1} = \frac{g_3^2}{N_2} = -2$ dB. (b) $\frac{g_1^2}{N_r} = \frac{g_1^2}{N_1} = 25$ dB, $\frac{g_2^2}{N_r} = \frac{g_2^2}{N_2} = 20$ dB and $\frac{g_3^2}{N_1} = \frac{g_3^2}{N_2} = 8$ dB.

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