On Degrees of Freedom of the Cognitive MIMO Two-Interfering Multiple-Access Channels

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***Abstract*—This paper characterizes the exact spatial Degrees of Freedom (DoF) region of the two-user multiple-input–multiple-output two-interfering multiple-access channels (2-IMACs), which models the uplink transmission in cellular networks, for three scenarios. The first scenario is the 2-IMAC without cognition, for which the achievability is generalized for more than two transmitters in each multiple-access channel. The two-user 2-IMACs' DoF regions are also derived for two different genie-aided cognition scenarios. Here, zero forcing by transmitters and interference alignment are studied in a network with more than two transmitters, whereas previous works dealt with two-transmitter networks. We also perform a comparative analysis on the DoF of the three scenarios.**

***Index Terms*—Degrees of Freedom (DoF) region, genie-aided cognitive radio (CR), interference alignment (IA), uplink channel.**

I. Introduction

New research directions in wireless communication are predominantly relevant to its exponentially accelerated throughput realized by cellular networks systems and services. Recent paradigms aimed at optimizing radio spectrum utilization reinforce interests in multiple-input–multiple-output (MIMO) systems, interference alignment (IA), and cognitive radio (CR) [1], [2].

In a MIMO system, transmission and reception of signals with multiple antennas provide an extra source of signal dimension other than frequency spectrum. This improvement introduces a multiplicative increase in the wireless throughput, which is measured by Degrees of Freedom (DoF; also known as capacity prelog factor or multiplexing gain). If the sum capacity of a network with real-valued channel coefficients can be written as $C_S(P_T) = (d/2)\log(P_T) + o(\log(P_T))$, where $\boldsymbol{P}_T$ denotes the sum of transmitted powers of all transmitters in the network, then $\boldsymbol{d}$ is the number of DoF. As a result, DoF provides an approximate characterization of $\boldsymbol{C}_S$ to within an $o(\log(\boldsymbol{P}_T))$ error, which diminishes as $\boldsymbol{P}_T$ increases and the network operates in the high SNR regime. The DoF region of a network also provides an approximation of the capacity region for high SNR values. Suppose there are $\boldsymbol{K}$ independent messages to be communicated in a network, and $\mathcal{C}(\boldsymbol{P}_T)$ is the capacity region of the network as a function of $\boldsymbol{P}_T$. The DoF region of such a network is defined as

$$\mathcal{D} \triangleq \Big\{ (d_1, \ldots, d_K) \in \mathcal{R}^{+^K} : \exists\,(R_1(P_T), \ldots, R_K(P_T)) \in \mathcal{C}(P_T), \text{ where } d_i = \lim_{P_T\to\infty} \frac{R_i(P_T)}{0.5\log(P_T)}, i = 1, \ldots, K \Big\}. \quad (1)$$

For a network with real-valued channel coefficients, $\log(\boldsymbol{P}_T)$ in denominator should be divided by 2.

Studying the DoF, as an approximation for Shannon capacity, of various communication channels is of fundamental importance and has drawn a lot of attention [3]–[13]. More investigations on the spatial DoF of the MIMO X channel resulted in the discovery of IA in [6] and [7]. This idea was then crystallized in [8] and [9] and proved to be a very powerful technique for achieving the optimal DoF of certain communication scenarios of interest. In spite of important practical issues yet to be resolved, IA offers a simple and elegant technique for interference management, which makes approaching the capacity in certain wireless networks possible as the SNR increases.

IA is a cooperative communication scheme through which we can utilize available signal space dimensions efficiently. The idea is that each transmitter transmits its messages in a way that they overlap and therefore occupy as few dimensions as possible, in the signal space of the nonintended receivers, while they remain resolvable in the signal space of the intended receiver. Thus, each transmitter implicitly cooperates with other nodes of the network through a "do not harm strategy" [13]. Theoretically, it is possible to align interference in the dimensions of time, frequency, or space in MIMO networks.

Using IA-based techniques, the exact characterization of the DoF region of the MIMO X channel was provided in [9], and it was shown that the MIMO X channel can assume noninteger optimal DoF. In [8], it was shown that the optimal sum DoF of the time-varying singe-input–single-output (SISO) $\boldsymbol{K}$-user interference channel (IC) is $\boldsymbol{K}/2$ asymptotically. This groundbreaking result revealed that "time-varying interference networks are not fundamentally interference limited" [8]. Introducing the concept of "subspace IA," the asymptotic DoF characterization of the SISO time-varying cellular channel was studied in [10]. In this paper, however, we deal with the exact characterization of the spatial DoF region of the uplink channel in MIMO cellular networks when the channel coefficients remain constant during communication.

In CR, cognitive users (CUs) are allowed to coexist, and perhaps cooperate, with non-cognitive or primary users (PUs)

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to realize optimal use of resources. This is in contrast with conventional schemes that exclusively offer spectrum to some predetermined licensed users. CUs are equipped to be aware of the communication environment. Then, using their environmental information and a strategy controlled by a predefined protocol, they maximize their own interest intelligently and at the same time minimize their interference to the licensed users [2].

In traditional proposals for CR protocols, the cognitive behavior is to determine the unused portions of the spectrum in either time or frequency, and "a device transmits over a certain time or frequency band only when no other user does" [14]. In an alternative approach, the licensed users and the CUs simultaneously transmit over the *same* time or frequency [14]. Here, cognition means that the message of the licensed users is presented to the CU noncausally by a "genie." Having the message of the PUs at hand, the CU adapts its transmission signal to achieve its own goal with the minimum possible interference to the PUs. The idea is expected to be more flexible and a potentially more efficient transmission model for the CR channels [14].

The genie-aided cognition, later called cognitive message sharing, has been studied from an information-theoretic point of view in various networks, e.g., the two-user X channel [9], [15] and IC [14], [16]. It is well known that when messages from some transmitters in the network are provided to some nodes other than the intended receiver(s), noncausally and through a noise-free channel, the DoF region will increase [2], [9], [15], [16]. These results, however, are presented for networks with two transmitters. Here, we extend such results and characterize the DoF region of a network with more than two transmitters.

This paper explores the benefits and limitations of genie-aided cognition in simple scenarios for uplink transmission in MIMO cellular networks at the high SNR regime. We characterize the DoF region of the two-user MIMO two-interfering multiple-access channels (2-IMACs) in three scenarios, i.e., when there is no cognition and for two genie-aided cognitive scenarios. The channel consists of four transmitters and two receivers, which model uplink transmission in a cellular network with two interfering cells and two users in each cell. The first cognitive scenario consists of a cognitive multiple-access channel (MAC) in close proximity of a primary MAC in which all messages of the PUs are shared with all CUs. In the second cognitive scenario, one of the transmitters in each MAC is cognitive while the other one is primary, and the message of the PU in a MAC is shared with the CU of the same MAC. The achievability schemes are based on linear precoding at the transmitter sides and linear zero forcing at the receiver side.

Our studies show that the MIMO cellular network is an example of multiuser communication channels that can assume noninteger optimal DoFs. This result holds not only for the no-cognition scenario but also for the two-user MIMO 2-IMACs under the first cognitive scenario. We will specifically prove that the maximum sum DoF is $4m/3$ and $3m/2$ for these two scenarios, respectively, where m is the number of antennas at all nodes. However, the total DoF for the second cognitive scenario will be equal to $2m$.

In contrast with the X-channel, in 2-IMACs, there is no broadcast component, i.e., all links are either intended or interference links. We observe that achieving the DoF region of the two-user 2-IMACs without cognition, despite the fact that its structure resembles the IC, needs IA as in the X-channel. It is well known that IA is not required to achieve optimal DoF of the two-user IC [4]. We will also show that if there is a cognitive transmitter in each MAC (the second cognitive scenario), IA is no longer required for achieving the DoF region.

The organization of this paper is as follows. Section II describes the system model. Section III characterizes the DoF region for the no message sharing scenario, and Sections IV and V deal with the DoF region of the two cognitive scenarios. Derived results are discussed in Section VI, and Section VII concludes this paper.

Here, the calligraphic letters $\mathcal{R}$ and $\mathcal{Z}$ represent the set of real and integer numbers, respectively. Bold italic capital letters stand for matrices, e.g., $\boldsymbol{H}$, and bold italic small letters represent vectors, e.g., $\boldsymbol{v}$. Normal italic letters are used for scalars, e.g., x. Finally, $convh(\mathcal{S})$ is the convex hull of the set $\mathcal{S}$, and $(x)^+$ stands for the function $\max(0, x)$.

II. System Model

This section introduces the three scenarios under study. Cognition in the systems is of the overlay paradigm [2], with genie-aided CR model, i.e., noncausal message sharing through a noise-free channel. For the no-cognition scenario, the achievability is also extended into a network with more than two transmitters in each MAC.

In all scenarios, m_j is the number of antennas at transmitter j for $j = 1, \ldots, 4$, and n_k is the number of antennas at receiver k for $k = 1, 2$. $\boldsymbol{H^{rk}} \in \mathcal{R}^{n_r \times m_k}$ represents the channel matrix between the kth transmitter and the rth receiver. It is assumed that the entries of all channel matrices are independently drawn from a continuous probability distribution, and therefore, all channel matrices are almost surely full rank. Here, the kth transmitter has an independent message s_k with d_k, $k = 1, \ldots, 4$, DoF to be communicated to its corresponding receiver. x_i^k represents the ith element $(i = 1, \ldots, d_k)$ of the codeword for the kth message. Each transmitter is subject to a power constraint of $\boldsymbol{P}_k$, $k = 1, \ldots, 4$, and $\boldsymbol{P}_T = \sum_{k=1}^4 \boldsymbol{P}_k$ is the sum of all transmitted powers.

A. No Message Sharing Scenario

In this scenario, we study two-user MIMO 2-IMACs without cognitive message sharing [refer to Fig. 1(a)]. In the first MAC, the kth $(k = 1, 2)$ transmitter sends s_k from its $\boldsymbol{m}_k$ antennas by transmitting x_i^k in the direction of $\boldsymbol{v_i^{kk}} \in \mathcal{R}^{m_k \times 1}$ beamforming vectors $(i = 1, \ldots, d_k)$ to receiver 1. The transmitters of the second MAC transmit their messages in the same way. Therefore, the received signal at receiver r is

$$\boldsymbol{y^r} = \sum_{j=1}^{4} \boldsymbol{H^{rj}} \left(\sum_{i=1}^{d_j} x_i^j \boldsymbol{v_i^{jj}} \right) + \boldsymbol{n^r}, r = 1, 2 \qquad (2)$$

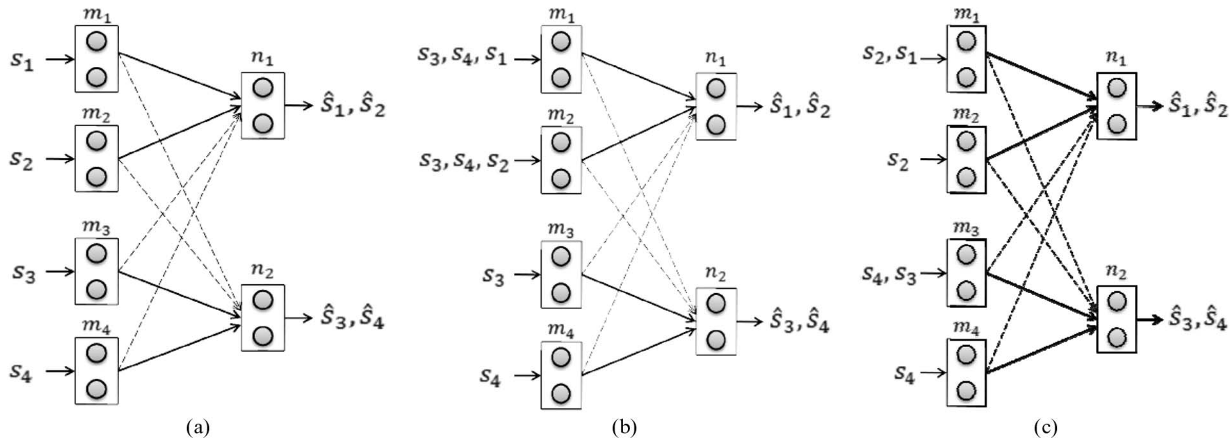

Fig. 1. Three scenarios under study. (a) No message sharing. (b) Cognitive MAC close to a primary MAC. (c) Cognitive transmitters in each MAC.

TABLE I
REQUIRED CSIT FOR DESIGNING BEAMFORMING VECTORS OF 2-IMAC WITHOUT COGNITION

Beam-forming vectors	are functions of
v_i^{jj}, for $j = 1, 2$, and $i = 1, \dots, d_j$	H^{2j} for $j = 1, 2$
v_i^{jj}, for $j = 3, 4$, and $i = 1, \dots, d_j$	H^{1j} for $j = 3, 4$

TABLE II
REQUIRED CSIT FOR BEAMFORMING VECTORS OF THE COGNITIVE MAC IN CLOSE PROXIMITY OF THE PRIMARY MAC SCENARIO

Beam-forming vectors	are functions of
v_i^{jj}, for $j = 1, 2$, and $i = 1, \dots, d_j$	H^{2j} for $j = 1, 2$
$\begin{bmatrix} v_i^{1k} \\ v_i^{2k} \end{bmatrix}$ and v_i^{kk}, for $k = 3, 4$, and $i = 1, \dots, d_k$	H^{1k} for $k = 1, \dots, 4$

where $\boldsymbol{n^r}$ is the additive white Gaussian noise (AWGN) vector at the rth receiver.

In the proposed achievability for this scenario, beamforming vectors are functions of channel matrices, as described in Table I.

B. Cognitive MAC in Close Proximity of a Primary MAC

In the second scenario, all transmitters of the first MAC (CUs) have noncasual access to all messages of the second MAC's transmitters [refer to Fig. 1(b)]. The kth PU $(k = 3, 4)$ sends s_k from its m_k antennas by transmitting x_i^k in the direction of $\boldsymbol{v_i^{kk}} \in \mathcal{R}^{m_k \times 1}$ beamforming vectors $(i = 1, \dots, d_k)$ to receiver 2. The jth CU $(j = 1, 2)$ sends s_j from its m_j antennas by transmitting x_i^j in the direction of $\boldsymbol{v_i^{jj}} \in \mathcal{R}^{m_j \times 1}$ beamforming vectors $(i = 1, \dots, d_j)$ to receiver 1. Furthermore, all CUs relay each of the PUs' messages, i.e., all CUs send the kth PU's message s_k, $k = 3, 4$, from $m_1 + m_2$ antennas by transmitting x_i^k in the direction of $\begin{bmatrix} \boldsymbol{v_i^{1k}} \\ \boldsymbol{v_i^{2k}} \end{bmatrix} \in \mathcal{R}^{(m_1+m_2)\times 1}$ beamforming vector $(i = 1, \dots, d_k)$ to the second receiver. Therefore, the received signal at receiver r is

$$\boldsymbol{y^r} = \sum_{j=1}^{2} \left(\boldsymbol{H}^{rj} \left(\sum_{i=1}^{d_j} x_i^j \boldsymbol{v_i^{jj}} + \sum_{l=3}^{4} \sum_{i=1}^{d_l} x_i^l \boldsymbol{v_i^{jl}} \right) \right) + \sum_{k=3}^{4} \left(\boldsymbol{H^{rk}} \sum_{i=1}^{d_k} x_i^k \boldsymbol{v_i^{kk}} \right) + \boldsymbol{n}^r, \quad r = 1, 2 \quad (3)$$

where $\boldsymbol{n^r}$ is the AWGN vector at receiver r.

In the proposed achievability for this scenario, beamforming vectors are functions of channel matrices, as described in Table II.

C. Cognitive Transmitters in Each MAC

In the third scenario, transmitters 1 and 3 are cognitive, and transmitters 2 and 4 are primary [refer to Fig. 1(c)]. The CU in a MAC has noncasual access to the message of the PU of the same MAC. In the first MAC, transmitter 2 sends s_2 from its m_2 antennas by transmitting x_i^2 in the direction of $\boldsymbol{v_i^{22}} \in \mathcal{R}^{m_2 \times 1}$ beamforming vectors $(i = 1, \dots, d_2)$ to the first receiver. Transmitter 1 sends s_1 from its m_1 antennas by transmitting x_i^1 in the direction of $\boldsymbol{v_i^{11}} \in \mathcal{R}^{m_1 \times 1}$ beamforming vectors $(i = 1, \dots, d_1)$. Moreover, it relays PU's message s_2 from its m_1 antennas by transmitting x_i^2 in the direction of $\boldsymbol{v_i^{12}} \in \mathcal{R}^{m_1 \times 1}$ beamforming vectors $(i = 1, \dots, d_2)$ for receiver 1. The transmitters of the second MAC share and transmit their messages in the same way. Therefore, the received signal at receiver r is

$$\boldsymbol{y}^r = \boldsymbol{H^{r1}} \left(\sum_{i=1}^{d_1} x_i^1 \boldsymbol{v_i^{11}} + \sum_{i=1}^{d_2} x_i^2 \boldsymbol{v_i^{12}} \right) + \boldsymbol{H}^{r2} \sum_{i=1}^{d_2} x_i^2 \boldsymbol{v_i^{22}} + \boldsymbol{H^{r3}} \left(\sum_{i=1}^{d_3} x_i^3 \boldsymbol{v_i^{33}} + \sum_{i=1}^{d_4} x_i^4 \boldsymbol{v_i^{34}} \right) + \boldsymbol{H^{r4}} \sum_{i=1}^{d_4} x_i^4 \boldsymbol{v_i^{44}} + \boldsymbol{n}^r, \quad r = 1, 2 \quad (4)$$

where $\boldsymbol{n^r}$ is the AWGN vector at receiver r.

TABLE III
REQUIRED CSIT FOR DESIGNING BEAMFORMING VECTORS FOR CUs IN EACH MAC SCENARIO

Beam-forming vectors	are functions of
v_i^{11}, for $i=1,\dots,d_1$, and v_i^{22}, and v_i^{12}, for $i=1,\dots,d_2$	H^{2j} for $j=1,2$
v_i^{33}, for $i=1,\dots,d_3$, and v_i^{44}, and v_i^{34}, for $i=1,\dots,d_4$	H^{1j} for $j=3,4$

Table III describes how beamforming vectors depend on channel matrices in the proposed achievability.

III. Two-Interfering Multiple-Access Channels Without Message Sharing

For the no-cognition scenario, the received signal at the rth receiver in (2) can be represented in the following way:

$$\boldsymbol{y^r} = \left[\boldsymbol{H^{r1}}\ \boldsymbol{H^{r2}}\ \boldsymbol{H^{r3}}\ \boldsymbol{H^{r4}}\right] \times \left(\sum_{i=1}^{d_1} x_i^1 \begin{bmatrix} \boldsymbol{v_i^{11}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \sum_{i=1}^{d_2} x_i^2 \begin{bmatrix} \mathbf{0} \\ \boldsymbol{v_i^{22}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \sum_{i=1}^{d_3} x_i^3 \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{v_i^{33}} \\ \mathbf{0} \end{bmatrix} + \sum_{i=1}^{d_4} x_i^4 \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{v_i^{44}} \end{bmatrix} \right) + \boldsymbol{n^r}, \quad r = 1, 2. \tag{5}$$

The following theorem establishes an outer bound for the DoF region.

Theorem 1: The outer bound on the DoF region of the two-user 2-IMACs with no message sharing is

$$\mathcal{D}_{\text{out}}^0 = \Big\{ (d_1, d_2, d_3, d_4) \in \mathcal{R}^{+^4} : d_1 + d_2 \le n_1, \\ d_3 + d_4 \le n_2, d_1 \le m_1, d_2 \le m_2, \\ d_3 \le m_3, d_4 \le m_4, d_1 + d_2 + d_3 \le \max(m_3, n_1), \\ d_1 + d_2 + d_4 \le \max(m_4, n_1), \\ d_2 + d_3 + d_4 \le \max(m_2, n_2), \\ d_1 + d_3 + d_4 \le \max(m_1, n_2), \\ \sum_{i=1}^{4} d_i \le \max(m_1 + m_2, n_2), \\ \sum_{i=1}^{4} d_i \le \max(m_3 + m_4, n_1) \Big\}. \tag{6}$$

Proof: The first six bounds are trivial bounds on the DoF of the MIMO multiple-access components of 2-IMAC. An IC [shown in Fig. 2(a)] can be obtained by omitting message s_4 from network and letting transmitters 1 and 2 share their messages. This message sharing will not reduce d_1, d_2, and d_3. Theorem 7 in [16] states that

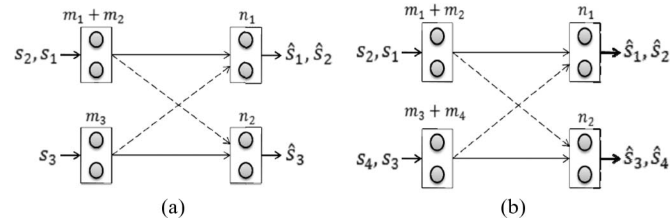

Fig. 2. ICs for the proof of Theorem 1.

$$d_1 + d_2 + d_3 \le \min\left\{ \sum_{i=1}^{3} m_i, n_1 + n_2, \max(m_1 + m_2, n_2), \max(m_3, n_1) \right\} \le \max(m_3, n_1). \tag{7}$$

This is the seventh bound. The eighth, ninth, and tenth bounds can be proved in a similar way.

The IC shown in Fig. 2(b) can be obtained by sharing messages between transmitters 1 and 2 and between transmitters 3 and 4. Using [16, Th. 7]

$$\sum_{i=1}^{4} d_i \le \min\left\{ \sum_{i=1}^{4} m_i, n_1 + n_2, \max(m_1 + m_2, n_2), \max(m_4 + m_3, n_1) \right\} \tag{8}$$

yields eleventh and twelfth bounds. ■

Before presenting the achievable DoF region of two-user 2-IMACs, we introduce the beamforming vector design ideas in the following example. This example shows how the achievability scheme can be generalized into 2-IMACs with more than two transmitters in each MAC. In a network with more than two interferers, the algorithm presented in Appendix B is used for aligning the interfering signals.

Example 1: For three-user MIMO 2-IMACs (transmitters 1, 2, and 3 are in the first MAC, and transmitters 4, 5, and 6 are in the second MAC) with $m_1 = 6$, $m_2 = 6$, $m_3 = 7$, $m_4 = 2$, $m_5 = 5$, and $m_6 = 7$ antennas at transmitters and $n_1 = 6$, $n_2 = 9$ antennas at receivers, DoFs $d_1 = 1$, $d_2 = 1$, $d_3 = 1$, $d_4 = 2$, $d_5 = 2$, and $d_6 = 4$ are achievable.

The received signal at the first receiver is

$$y^1 = \left[\boldsymbol{H^{11}}\ \boldsymbol{H^{12}} \boldsymbol{H^{13}} \boldsymbol{H^{14}} \boldsymbol{H^{15}} \boldsymbol{H^{16}}\right] \times \left(x_1^1 \begin{bmatrix} \boldsymbol{v_1^{11}} \\ \mathbf{0} \end{bmatrix} + x_1^2 \begin{bmatrix} \mathbf{0} \\ \boldsymbol{v_1^{22}} \\ \mathbf{0} \end{bmatrix} + x_1^3 \begin{bmatrix} \mathbf{0} \\ \boldsymbol{v_1^{33}} \\ \mathbf{0} \end{bmatrix} + \sum_{i=1}^{2} x_i^4 \begin{bmatrix} \mathbf{0} \\ \boldsymbol{v_i^{44}} \\ \mathbf{0} \end{bmatrix} + \sum_{i=1}^{2} x_i^5 \begin{bmatrix} \mathbf{0} \\ v_i^{\mathbf{55}} \\ \mathbf{0} \end{bmatrix} + \sum_{i=1}^{4} x_i^6 \begin{bmatrix} \mathbf{0} \\ v_i^{\mathbf{66}} \end{bmatrix} \right) + \boldsymbol{n^1} \tag{9}$$

with s_4, s_5, and s_6 as interfering messages. Since $(m_6 - n_1)^+ = 1$, we can select beamforming vector $\boldsymbol{v_1^{66}} \in \mathcal{R}^7$ as the basis of the null space of $\boldsymbol{H^{16}}$ for zero forcing by the transmitter. Then, using the algorithm of Appendix B, we design the rest of the beamforming vectors to be aligned at the first receiver. According to Lemma A.1, for the dimension of the intersection subspace between column spaces of $\boldsymbol{H}^{\mathbf{14}}$, $\boldsymbol{H}^{\mathbf{15}}$, and $\boldsymbol{H}^{\mathbf{16}}$ using (A.1), we have

$$\begin{aligned} C_{\{4,5,6\}} &= \big(m_4+m_5+m_6-2n_1-(m_4-n_1)^+ \\ &\qquad -(m_5-n_1)^+-(m_6-n_1)^+\big)^+=1 \\ C_{\{4,5\}} &= \big(m_4+m_5-n_1-(m_4-n_1)^+-(m_5-n_1)^+\big)^+=1 \\ C_{\{4,6\}} &= \big(m_4+m_6-n_1-(m_4-n_1)^+-(m_6-n_1)^+\big)^+=2 \\ C_{\{5,6\}} &= \big(m_5+m_6-n_1-(m_5-n_1)^+-(m_6-n_1)^+\big)^+=5. \end{aligned}$$

Since $C_{\{4,5,6\}} = 1$ and $r_3 = 1$, we can design $\boldsymbol{v}_{\mathbf{1}}^{\mathbf{44}} \in \mathcal{R}^2$, $\boldsymbol{v}_{\mathbf{1}}^{\mathbf{55}} \in \mathcal{R}^5$ and $\boldsymbol{v}_{\mathbf{2}}^{\mathbf{66}} \in \mathcal{R}^7$ in a way that they become aligned along one dimension. For the design of the remaining beamforming vectors, since $C_{\{4,5\}} - C_{\{4,5,6\}} = 0$, the interference of messages s_4 and s_5 can no longer be aligned. However, since $C_{\{4,6\}} - C_{\{4,5,6\}} = 1$, we can design $\boldsymbol{v}_{\mathbf{2}}^{\mathbf{44}} \in \mathcal{R}^2$ and $\boldsymbol{v}_{\mathbf{3}}^{\mathbf{66}} \in \mathcal{R}^7$ for alignment. Furthermore, since $C_{\{5,6\}} - C_{\{4,5,6\}} = 4$, it is possible to design $\boldsymbol{v}_{\mathbf{2}}^{\mathbf{55}} \in \mathcal{R}^5$ and $\boldsymbol{v}_{\mathbf{4}}^{\mathbf{66}} \in \mathcal{R}^7$ for alignment. Thus, at the first receiver, the range space of these three interfering signals is 3-dimensional; the receiver will not observe the vector $\boldsymbol{H}^{\mathbf{16}}\boldsymbol{v}_{\mathbf{1}}^{\mathbf{66}} = 0$, and each of following vectors

$$\begin{aligned} \boldsymbol{q}^{\mathbf{1}} &= \boldsymbol{H}^{\mathbf{14}}\boldsymbol{v}_{\mathbf{1}}^{\mathbf{44}} = \boldsymbol{H}^{\mathbf{15}}\boldsymbol{v}_{\mathbf{1}}^{\mathbf{55}} = \boldsymbol{H}^{\mathbf{16}}\boldsymbol{v}_{\mathbf{2}}^{\mathbf{66}} \\ \boldsymbol{q}^{\mathbf{2}} &= \boldsymbol{H}^{\mathbf{14}}\boldsymbol{v}_{\mathbf{2}}^{\mathbf{44}} = \boldsymbol{H}^{\mathbf{16}}\boldsymbol{v}_{\mathbf{3}}^{\mathbf{66}} \\ \boldsymbol{q}^{\mathbf{3}} &= \boldsymbol{H}^{\mathbf{15}}\boldsymbol{v}_{\mathbf{1}}^{\mathbf{55}} = \boldsymbol{H}^{\mathbf{16}}\boldsymbol{v}_{\mathbf{4}}^{\mathbf{66}} \end{aligned}$$

occupy one dimension. The three remaining dimensions are reserved for the intended signals.

The received signal at the second receiver is

$$\begin{aligned} \boldsymbol{y}^2 &= \left[\boldsymbol{H}^{\mathbf{21}}\ \boldsymbol{H}^{\mathbf{22}}\ \boldsymbol{H}^{\mathbf{23}}\ \boldsymbol{H}^{\mathbf{24}}\ \boldsymbol{H}^{\mathbf{25}}\ \boldsymbol{H}^{\mathbf{26}}\right] \\ &\times\Bigg(x_1^1\begin{bmatrix}\boldsymbol{v}_{\mathbf{1}}^{\mathbf{11}}\\ \mathbf{0}\end{bmatrix}+x_1^2\begin{bmatrix}\mathbf{0}\\ \boldsymbol{v}_{\mathbf{1}}^{\mathbf{22}}\\ \mathbf{0}\end{bmatrix}+x_1^3\begin{bmatrix}\mathbf{0}\\ \boldsymbol{v}_{\mathbf{1}}^{\mathbf{33}}\\ \mathbf{0}\end{bmatrix} \\ &\quad+\sum_{i=1}^{2}x_i^4\begin{bmatrix}\mathbf{0}\\ \boldsymbol{v}_{\boldsymbol{i}}^{\mathbf{44}}\\ \mathbf{0}\end{bmatrix}+\sum_{i=1}^{2}x_i^5\begin{bmatrix}\mathbf{0}\\ \boldsymbol{v}_{\boldsymbol{i}}^{\mathbf{55}}\\ \mathbf{0}\end{bmatrix}+\sum_{i=1}^{4}x_i^6\begin{bmatrix}\mathbf{0}\\ \boldsymbol{v}_{\boldsymbol{i}}^{\mathbf{66}}\end{bmatrix}\Bigg)+\boldsymbol{n}^2 \end{aligned} \tag{10}$$

with s_1, s_2, and s_3 as interfering signals. Since, according to (A.1)

$$\begin{aligned} C_{\{1,2,3\}} = \big(m_1 + m_2 + m_3 - 2n_2 - (m_1 - n_2)^+ \\ -(m_2 - n_2)^+ - (m_3 - n_2)^+\big)^+ = 1 \end{aligned}$$

we can design $\boldsymbol{v}_{\mathbf{1}}^{\mathbf{11}} \in \mathcal{R}^6$, $\boldsymbol{v}_{\mathbf{1}}^{\mathbf{22}} \in \mathcal{R}^6$, and $\boldsymbol{v}_{\mathbf{1}}^{\mathbf{33}} \in \mathcal{R}^7$ beamforming vectors in a way that they are aligned along one dimension. The eight remaining dimensions at the second receiver are assigned to intended signals (s_4, s_5, and s_6).

It can be verified that these DoF values are in the boundary of the DoF region. If we omit messages s_1 and s_2 in the network, and all transmitters of the second MAC share their messages, the IC shown in Fig. 3 will be obtained. According to [16], we have $d_3 + d_4 + d_5 + d_6 \leq 9$. Similarly, $d_2 + d_4 + d_5 + d_6 \leq 9$ and $d_1 + d_4 + d_5 + d_6 \leq 9$.

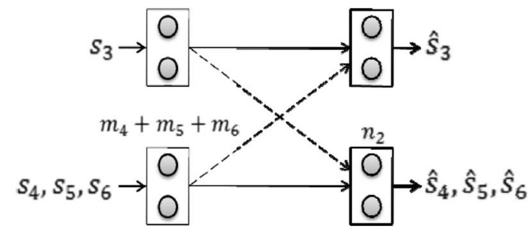

Fig. 3. ICs for the upper bound of Example 1.

Now, we continue with characterizing the DoF region of two-user MIMO 2-IMACs.

Theorem 2: For the two-user 2-IMAC with no message sharing, an achievable DoF region is

$$\begin{aligned} \mathcal{D}_{\text{in}}^0 &\triangleq conv h\left(\mathcal{D}_{\text{out}}^0 \cap \mathcal{Z}^{+^4}\right) \\ &= conv h\Big(\Big\{(d_1, d_2, d_3, d_4) \in \mathcal{Z}^{+^4} : \\ &\qquad d_1 \leq m_1, d_2 \leq m_2, d_3 \leq m_3, d_4 \leq m_4 \\ &\qquad I_2 + d_3 + d_4 \leq n_2, I_1 + d_1 + d_2 \leq n_1\Big\}\Big) \end{aligned} \tag{11}$$

where

$$\begin{aligned} I_2 \triangleq\, &\big(d_1-(m_1-n_2)^+\big)^+ + \big(d_2-(m_2-n_2)^+\big)^+ \\ &-\min\Big\{\big(d_1-(m_1-n_2)^+\big)^+, \big(d_2-(m_2-n_2)^+\big)^+ \\ &\qquad \big(m_1+m_2-n_2-(m_1-n_2)^+-(m_2-n_2)^+\big)^+\Big\} \end{aligned} \tag{12}$$

$$\begin{aligned} I_1 \triangleq\, &\big(d_3-(m_3-n_1)^+\big)^+ + \big(d_4-(m_4-n_1)^+\big)^+ \\ &-\min\Big\{\big(d_3-(m_3-n_1)^+\big)^+, \big(d_4-(m_4-n_1)^+\big)^+ \\ &\qquad \big(m_3+m_4-n_1-(m_3-n_1)^+-(m_4-n_1)^+\big)^+\Big\}. \end{aligned} \tag{13}$$

Proof: The achievability scheme is based on the techniques used for the two-user X channel in [9]. Based on (5), we need to design a set of linear beamforming vectors for each message, i.e.,

$$\begin{aligned} \mathcal{V}^1 &= \left\{\begin{bmatrix}\boldsymbol{v}_{\boldsymbol{i}}^{\mathbf{11}}\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1}, i = 1, \ldots, d_1\right\} \\ \mathcal{V}^2 &= \left\{\begin{bmatrix}\mathbf{0}\\ \boldsymbol{v}_{\boldsymbol{i}}^{\mathbf{22}}\\ \mathbf{0}\\ \mathbf{0}\end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1}, i = 1, \ldots, d_2\right\} \\ \mathcal{V}^3 &= \left\{\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \boldsymbol{v}_{\boldsymbol{i}}^{\mathbf{33}}\\ \mathbf{0}\end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1}, i = 1, \ldots, d_3\right\} \\ \mathcal{V}^4 &= \left\{\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ \boldsymbol{v}_{\boldsymbol{i}}^{\mathbf{44}}\end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1}, i = 1, \ldots, d_4\right\}. \end{aligned} \tag{14}$$

At the receiver, the set of vectors carrying intended signals is required to be resolvable, i.e., they must be linearly independent. Whereas, the vectors that carry interference do not need to be resolvable; a number of such vectors may be aligned along one received vector. To achieve optimal DoF by linear zero forcing, at the signal space of a receiver, all received vectors must be linearly independent, as discussed in lemma 14 in [16]. Therefore, it is necessary that the union of $\bigcup_{i=1}^{4} \mathcal{V}^i$ be a set of linearly independent vectors, which is guaranteed by the first four bounds in $\mathcal{D}_{\text{in}}^0$.

At the second receiver, s_1 and s_2 are interfering messages. The aim is to design $d_1 + d_2$ vectors of the sets $\mathcal{V}^1$ and $\mathcal{V}^2$ such that they occupy as few dimensions as possible in the signal space of the second receiver while remaining linearly independent at the intended receiver, i.e., they occupy a $(d_1 + d_2)$-dimensional subspace in the signal space of the first receiver. To this end, we first design the beamforming vectors for zero forcing the interfering signals by transmitters. Subsequently, we use IA to design the possible remaining vectors.

To zero force the interfering signals by transmitters, we select the first $(m_1 - n_2)^+$ vectors of $\mathcal{V}^1$ as the basis of the null space of $\boldsymbol{H^{21}}$ and the first $(m_2 - n_2)^+$ vectors of $\mathcal{V}^2$ as the basis of the null space of $\boldsymbol{H^{22}}$. We can design other possible vectors of $\mathcal{V}^1$ and $\mathcal{V}^2$ to be aligned at the second receiver. According to Lemma A.1, in (A.1), the dimension of the intersection subspace between the column spaces of $\boldsymbol{H^{21}}$ and $\boldsymbol{H^{22}}$ is $C_{\{1,2\}} = (m_1 + m_2 - n_2 - (m_1 - n_2)^+ - (m_2 - n_2)^+)^+$. After performing zero forcing by transmitters, we can align at most $C_{\{1,2\}}$ dimensions of the interfering signals. If more beamforming vectors are required, i.e., $d_1 > (m_1 - n_2)^+ + C_{\{1,2\}}$ or $d_2 > (m_2 - n_2)^+ + C_{\{1,2\}}$, we generate them at random from a continuous probability distribution, guaranteeing linear independency of these vectors with high probability. Therefore, the dimension of the subspace spanned by the interference at the second receiver is given by I_2 in (12).

In the n_2-dimensional space of the second receiver, the dimension of the subspace spanned by the received interference is I_2, and the dimension of the subspace spanned by the desired signals is $d_3 + d_4$. To use the linear zero-forcing receiver, all $I_2 + d_3 + d_4$ received vectors must be linearly independent, which is guaranteed by the fifth bound in $\mathcal{D}_{\text{in}}^0$. Beamforming vectors $\mathcal{V}^3$ and $\mathcal{V}^4$ can be designed in a similar way.

Consequently, all integer-valued DoFs (d_1, d_2, d_3, d_4) that satisfy all six bounds of $\mathcal{D}_{\text{in}}^0$ and the convex hull of all such points (by time sharing) are achievable. In part D.1 of Appendix D, we show that every integer-valued point in the outer bound $\mathcal{D}_{\text{out}}^0$ is achievable by the aforementioned scheme. This completes the proof. ■

Since the achievable region $\mathcal{D}_{\text{in}}^0$ in (11) is the convex hull of all integer points in the outer bound of $\mathcal{D}_{\text{out}}^0$ in (6), if a corner point of the outer bound assumes noninteger values, there will be a gap between the inner and outer bounds. As an example, if there are m antennas in all nodes, then $(d_1 = m/3, d_2 = m/3, d_3 = m/3, d_4 = m/3)$ is a corner point of $\mathcal{D}_{\text{out}}^0$, which is noninteger if m is not a multiple of three, and therefore, it is not achievable by Theorem 2. However, in the following example, we will observe that the noninteger spatial DoFs are also achievable by applying the scheme introduced in Theorem 2 for the multiletter extension of the channel.

Example 2: In the two-user 2-IMAC with no message sharing with $m > 1$ antennas at all nodes, $(d_1 = m/3, d_2 = m/3, d_3 = m/3, d_4 = m/3)$ is almost surely achievable by three-symbol extension of the constant channel.

If we use this channel three times, the received signal at receiver r can be written in the following way:

$$\overline{\boldsymbol{y^r}} = \left[\overline{\boldsymbol{H^{r1}}}\ \overline{\boldsymbol{H^{r2}}}\ \overline{\boldsymbol{H^{r3}}}\ \overline{\boldsymbol{H^{r4}}}\right] \times \left(\sum_{i=1}^{m} x_i^1 \begin{bmatrix} \overline{\boldsymbol{v_i^{11}}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + \sum_{i=1}^{m} x_i^2 \begin{bmatrix} \boldsymbol{0} \\ \overline{\boldsymbol{v_i^{22}}} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + \sum_{i=1}^{m} x_i^3 \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \overline{\boldsymbol{v_i^{33}}} \\ \boldsymbol{0} \end{bmatrix} + \sum_{i=1}^{m} x_i^4 \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \overline{\boldsymbol{v_i^{44}}} \end{bmatrix} \right) + \overline{\boldsymbol{n^r}}, \quad r = 1, 2 \tag{15}$$

where $\overline{\boldsymbol{H^{ri}}} \in \mathcal{R}^{3m \times 3m}$ is the block diagonal channel matrix that represents the channel between the rth receiver and the ith transmitter in the three-symbol extension model

$$\overline{\boldsymbol{H^{ri}}} \triangleq \begin{bmatrix} \boldsymbol{H^{ri}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H^{ri}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H^{ri}} \end{bmatrix}. \tag{16}$$

Matrices $\boldsymbol{H^{ri}}$ are assumed to be full rank and remain constant during communication. $\overline{\boldsymbol{v_i^{jj}}} \in \mathcal{R}^{3m \times 1}$ is the beamforming vector for the ith element of the codeword for message j in the three-symbol extension model, i.e., $\boldsymbol{v_i^{jj}}(t)$ is the beamforming vector for the ith element of the codeword for message j at time $t = 1, 2, 3$

$$\overline{\boldsymbol{v_i^{jj}}} \triangleq \begin{bmatrix} \boldsymbol{v_i^{jj}}(1) \\ \boldsymbol{v_i^{jj}}(2) \\ \boldsymbol{v_i^{jj}}(3) \end{bmatrix} \tag{17}$$

and $\overline{\boldsymbol{n^r}}$ is the AWGN vector in the extended channel at the rth receiver.

For $m > 1$, to design the set of beamforming vectors for each message

$$\mathcal{V}^1 = \left\{ \begin{bmatrix} \overline{\boldsymbol{v_i^{11}}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \in \mathcal{R}^{12m \times 1}, i = 1, \ldots, m \right\}$$
$$\mathcal{V}^2 = \left\{ \begin{bmatrix} \boldsymbol{0} \\ \overline{\boldsymbol{v_i^{22}}} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \in \mathcal{R}^{12m \times 1}, i = 1, \ldots, m \right\}$$
$$\mathcal{V}^3 = \left\{ \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \overline{\boldsymbol{v_i^{33}}} \\ \boldsymbol{0} \end{bmatrix} \in \mathcal{R}^{12m \times 1}, i = 1, \ldots, m \right\}$$
$$\mathcal{V}^4 = \left\{ \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \overline{\boldsymbol{v_i^{44}}} \end{bmatrix} \in \mathcal{R}^{12m \times 1}, i = 1, \ldots, m \right\} \tag{18}$$

we use the scheme proposed in Theorem 2 for the three-symbol extension channel model of (15). Thus, $(\overline{d_1} = m, \overline{d_2} = m, \overline{d_3} = m, \overline{d_4} = m)$ is achievable, and $(d_1 = m/3, d_2 = m/3, d_3 = m/3, d_4 = m/3)$ is the DoF per channel use.

In a network with $m = 1$ (a SISO network), however, all the channel matrices $\overline{\boldsymbol{H}^{ri}}$ in (15) are a multiple of the 3×3 identity matrix. Subsequently, if we design some beamforming vectors to be aligned at the nonintended receiver, they will also be aligned at the intended receivers, and the scheme of Theorem 2 is not applicable. Since in this paper our aim is to characterize the spatial DoF region of the MIMO channel, we do not deal with the DoF of the SISO system.

The following theorem establishes the exact characterization of the spatial DoF region.

Theorem 3: $\mathcal{D}^0 = \mathcal{D}^0_{\text{out}}$.

Proof: Using the concept of normalized DoF, with an argument similar to [9, Lemmas 2 and 3 and Th. 11], it follows that $\mathcal{D}^0_{\text{out}}$ is the exact characterization of the DoF region. ■

IV. Cognitive Multiple-Access Channel in Close Proximity of a Primary Multiple-Access Channel

For the channel described in Section II-B, the received signals at the rth receiver in (3) can be represented in the following way:

$$\boldsymbol{y^r} = \left[\boldsymbol{H^{r1}}\ \boldsymbol{H^{r2}}\ \boldsymbol{H^{r3}}\ \boldsymbol{H^{r4}}\right] \times \left(\sum_{i=1}^{d_1} x_i^1 \begin{bmatrix} \boldsymbol{v_i^{11}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + \sum_{i=1}^{d_2} x_i^2 \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{v_i^{22}} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + \sum_{i=1}^{d_3} x_i^3 \begin{bmatrix} \boldsymbol{v_i^{13}} \\ \boldsymbol{v_i^{23}} \\ \boldsymbol{v_i^{33}} \\ \boldsymbol{0} \end{bmatrix} + \sum_{i=1}^{d_4} x_i^4 \begin{bmatrix} \boldsymbol{v_i^{14}} \\ \boldsymbol{v_i^{24}} \\ \boldsymbol{0} \\ \boldsymbol{v_i^{44}} \end{bmatrix} \right) + \boldsymbol{n^r}, \quad r = 1, 2. \tag{19}$$

In the following theorem, we establish an outer bound for the DoF region.

Theorem 4: The outer bound on the DoF region of the two-user 2-IMACs for the cognitive scenario described in Section II-B is

$$\begin{aligned} \mathcal{D}^1_{\text{out}} = \Big\{ &(d_1, d_2, d_3, d_4) \in \mathcal{R}^{+^4} : d_1 + d_2 \le n_1 \\ &d_3 + d_4 \le n_2, d_1 \le m_1, d_2 \le m_2 \\ &\sum_{i=1}^{4} d_i \le \sum_{i=1}^{4} m_i, d_1 + d_2 + d_3 \le m_1 + m_2 + m_3 \\ &d_1 + d_2 + d_4 \le m_1 + m_2 + m_4 \\ &d_2 + d_3 + d_4 \le \max(m_2, n_2) \\ &d_1 + d_3 + d_4 \le \max(m_1, n_2) \\ &\sum_{i=1}^{4} d_i \le \max(m_1 + m_2, n_2) \Big\}. \end{aligned} \tag{20}$$

Proof: The first seven bounds are trivial bounds on the DoF of the MIMO multiple-access and broadcast channel components of the 2-IMACs with message sharing. For the eighth bound, we obtain the IC with one message sharing, shown in Fig. 4(a); if we omit message s_1 from the channel, the third and fourth transmitters also share their messages. Of course, sharing messages s_3 and s_4 will not decrease the DoF region of messages s_2, s_3, and s_4. Then, from [16]

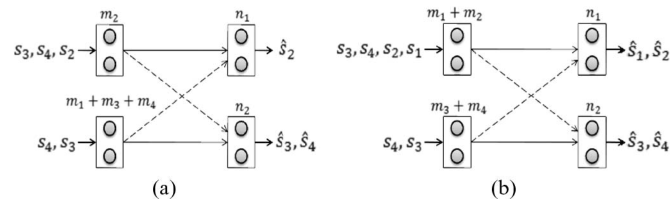

Fig. 4. ICs with one message sharing for the proof of Theorem 4.

$$d_2 + d_3 + d_4 \le \min\left\{ \sum_{i=1}^{4} m_i, n_1 + n_2, \max(m_2, n_2) \right\} \le \max(m_2, n_2). \tag{21}$$

The ninth bound is derived in a similar way. The tenth bound is derived by sharing s_1 and s_2 between the first and second transmitters and sharing s_3 and s_4 between the third and forth transmitters to obtain an IC with one message sharing, as shown in Fig. 4(b). Consequently, [16, Th. 7] yields the outer bound. ■

To establish the achievable DoF region for this scenario, we make use of the following theorem. (Proof is given in Appendix C.)

Theorem 5: Consider matrices $[\boldsymbol{A} \quad \boldsymbol{B^1}]$ and $[\boldsymbol{A} \quad \boldsymbol{B^2}]$. All entries of $\boldsymbol{A} \in \mathcal{R}^{n\times m}$, $\boldsymbol{B^1} \in \mathcal{R}^{n\times m_1}$, and $\boldsymbol{B^2} \in \mathcal{R}^{n\times m_2}$ are independently drawn from a continuous probability distribution, so these matrices and the concatenated matrices ($[\boldsymbol{A} \quad \boldsymbol{B^1}]$ and $[\boldsymbol{A} \quad \boldsymbol{B^2}]$) are almost surely full rank. If $[\boldsymbol{A} \quad \boldsymbol{B^1}]\, \boldsymbol{N_1} = [\boldsymbol{A} \quad \boldsymbol{B^1}] \begin{bmatrix} \boldsymbol{N_{1u}} \\ \boldsymbol{N_{1d}} \end{bmatrix} = \boldsymbol{0}$ and $[\boldsymbol{A} \quad \boldsymbol{B^2}]\boldsymbol{N_2} = [\boldsymbol{A} \quad \boldsymbol{B^2}] \begin{bmatrix} \boldsymbol{N_{2u}} \\ \boldsymbol{N_{2d}} \end{bmatrix} = \boldsymbol{0}$, where $\boldsymbol{N_{1u}} \in \mathcal{R}^{m\times(m\,+\,m_1\,-\,n)^+}$, $\boldsymbol{N_{1d}} \in \mathcal{R}^{m_1\times(m+m_1\,-\,n)^+}$, $\boldsymbol{N_{2u}} \in \mathcal{R}^{m\times(m\,+\,m_2\,-\,n)^+}$, and $\boldsymbol{N_{2d}} \in \mathcal{R}^{m_2\times(m+m_2-n)^+}$, then the number of independent columns in

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{N_{1u}} & \boldsymbol{N_{2u}} \\ \boldsymbol{N_{1d}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{N_{2d}} \end{bmatrix} \tag{22}$$

is given by

$$f_1 = (m + m_1 - n)^+ + (m + m_2 - n)^+ - (m - n)^+. \tag{23}$$

We proceed with deriving an achievable DoF region in the following theorem.

Theorem 6: An achievable DoF region for the scenario described in Section II-B is

$$\begin{aligned} \mathcal{D}^1_{\text{in}} &\triangleq conv h(\mathcal{D}^1_{\text{out}} \cap \mathcal{Z}^{+^4}) \\ &= conv h\Big(\Big\{ (d_1, d_2, d_3, d_4) \in \mathcal{Z}^{+^4} : d_1 \le m_1, d_2 \le m_2 \\ &\qquad d_3 \le m_1 + m_2 + m_3, d_4 \le m_1 + m_2 + m_4 \\ &\qquad \sum_{i=1}^{4} d_i \le \sum_{i=1}^{4} m_i, I_2 + d_3 + d_4 \le n_2, \\ &\qquad I_1 + d_1 + d_2 \le n_1 \Big\} \Big) \end{aligned} \tag{24}$$

where

$$I_2 \triangleq \left(d_1-(m_1-n_2)^+\right)^+ + \left(d_2-(m_2-n_2)^+\right)^+ - \min\left\{\left(d_1-(m_1-n_2)^+\right)^+, \left(d_2-(m_2-n_2)^+\right)^+, \left(m_1+m_2-n_2-(m_1-n_2)^+ - (m_2-n_2)^+\right)^+\right\}, \quad (25)$$

$$I_1 \triangleq \begin{cases} X_3+X_4-\min\{X_3, X_4, A\} & \text{if } m_1+m_2<n_1 \\ ((d_3-m_3)^+ + (d_4-m_4)^+ \\ -(m_1+m_2-n_1))^+ & \text{if } m_1+m_2\geq n_1 \end{cases} \quad (26)$$

where

$$A \triangleq \left(\sum_{i=1}^{4} m_i - n_1 - (m_1+m_2+m_3-n_1)^+ - (m_1+m_2+m_4-n_1)^+\right)^+ \quad (27)$$

$$X_3 \triangleq \left(d_3 - (m_1+m_2+m_3-n_1)^+\right)^+ \quad (28)$$

$$X_4 \triangleq \left(d_4 - (m_1+m_2+m_4-n_1)^+\right)^+. \quad (29)$$

Proof: In a similar manner to Theorem 2, based on (19), we need to design

$$\mathcal{V}^1 = \left\{ \begin{bmatrix} \boldsymbol{v}_i^{11} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1}, i = 1, \ldots, d_1 \right\}$$

$$\mathcal{V}^2 = \left\{ \begin{bmatrix} \mathbf{0} \\ \boldsymbol{v}_i^{22} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1}, i = 1, \ldots, d_2 \right\}$$

$$\mathcal{V}^3 = \left\{ \begin{bmatrix} \boldsymbol{v}_i^{13} \\ \boldsymbol{v}_i^{23} \\ \boldsymbol{v}_i^{33} \\ \mathbf{0} \end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1}, i = 1, \ldots, d_3 \right\}$$

$$\mathcal{V}^4 = \left\{ \begin{bmatrix} \boldsymbol{v}_i^{14} \\ \boldsymbol{v}_i^{24} \\ \mathbf{0} \\ \boldsymbol{v}_i^{44} \end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1}, i = 1, \ldots, d_4 \right\}. \quad (30)$$

By the same reasoning as in the proof of Theorem 2, all vectors in the union of $\bigcup_{i=1}^4 \mathcal{V}^i$ must be linearly independent, which is guaranteed by the first five bounds in $\mathcal{D}_{\text{in}}^1$.

At the second receiver, s_1 and s_2 are interfering messages. The design of beamforming vectors in $\mathcal{V}^1$ and $\mathcal{V}^2$ is as presented in the proof of Theorem 2. At the first receiver, s_3 and s_4 are interfering messages. Using Theorem 5, we first design the beamforming vectors for zero forcing the interfering signals by transmitters. Then, we make use of IA for designing the remaining vectors. According to Theorem 5, we should consider two different cases: $m_1+m_2<n_1$ and $m_1+m_2\geq n_1$.

In case of $m_1+m_2<n_1$, we can select the first $(m_1+m_2+m_3-n_1)^+$ vectors of $\mathcal{V}^3$ as the basis of the null space of $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{13}]$ and the first $(m_1+m_2+m_4-n_1)^+$ vectors of $\mathcal{V}^4$ as the basis of the null space of $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{14}]$. Since $m_1+m_2<n_1$, these vectors are linearly independent according to Theorem 5. If more beamforming vectors are required, i.e., $X_3>0$ and $X_4>0$, we may be able to design these vectors to be aligned at the first receiver. According to Lemma A.1 in Appendix A, considering matrices $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{13}]$ and $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{14}]$, we can select $(m_1+m_2+\sum_{i=1}^4 m_i - n_1 - (m_1+m_2+m_3-n_1)^+ - (m_1+m_2+m_4-n_1)^+)^+$ vectors for alignment. However, it can be verified that only the first A, defined in (27), beamforming vectors are linearly independent. Since the difference of each pair of beamforming vectors designed for alignment lies in the null space of $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{13}\ \boldsymbol{H}^{14}]$, which is $(\sum_{i=1}^4 m_i - n_1)^+$–dimensional

$$[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{13}] \begin{bmatrix} \boldsymbol{v}_i^{13} \\ \boldsymbol{v}_i^{23} \\ \boldsymbol{v}_i^{33} \\ \mathbf{0} \end{bmatrix} = [\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{14}] \begin{bmatrix} \boldsymbol{v}_i^{14} \\ \boldsymbol{v}_i^{24} \\ \mathbf{0} \\ \boldsymbol{v}_i^{44} \end{bmatrix}$$

$$\Rightarrow [\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{13}\ \boldsymbol{H}^{14}] \begin{bmatrix} \boldsymbol{v}_i^{13} - \boldsymbol{v}_i^{14} \\ \boldsymbol{v}_i^{23} - \boldsymbol{v}_i^{24} \\ \boldsymbol{v}_i^{33} \\ -\boldsymbol{v}_i^{44} \end{bmatrix} = \mathbf{0} \quad (31)$$

after zero forcing by transmitters only A linearly independent beamforming vectors are left for alignment. Thus, we can align $\min\{X_3, X_4, A\}$ vectors at the first receiver. If still more beamforming vectors are required, i.e., $d_3 > (m_1+m_2+m_3-n_1)^+ + \min\{X_3, X_4, A\}$ or $d_4 > (m_1+m_2+m_4-n_1)^+ + \min\{X_3, X_4, A\}$, we generate them at random from a continuous probability distribution to almost surely guarantee their linear independency. Therefore, the dimension of the subspace spanned by the interference at the first receiver is given by the first expression of I_1 in (26).

In case of $m_1+m_2\geq n_1$, we have $(m_1+m_2+m_3-n_1)^+$ vectors as the basis of the null space of $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{13}]$ and $(m_1+m_2+m_4-n_1)^+$ vectors as the basis of the null space of $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{14}]$. However, according to Theorem 5, these two null spaces have $m_1+m_2-n_1$ common dimensions. We can demonstrate the number of possible beamforming vectors for zero forcing the interference by transmitters in the Venn diagram of Fig. 5.

In this case, we can select the first m_3 vectors of $\mathcal{V}^3$ as the basis of the null space of $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{13}]$ and the first m_4 vectors of $\mathcal{V}^4$ as the basis of the null space of $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{14}]$. If $d_3>m_3$ or $d_4>m_4$, it is possible to select $m_1+m_2-n_1$ more beamforming vectors either from the null space of $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{13}]$ for $\mathcal{V}^3$ or from null space of $[\boldsymbol{H}^{11}\ \boldsymbol{H}^{12}\ \boldsymbol{H}^{14}]$ for $\mathcal{V}^4$ or a combination of $m_1+m_2-n_1$ such vectors for $\mathcal{V}^3$ and $\mathcal{V}^4$. It should be noted that by selecting more beamforming vectors for zero forcing, according to Theorem 5, the union of $\mathcal{V}^3\cup\mathcal{V}^4$ can no longer be a set of linearly independent vectors. In the rest of the proof, we will see that in case $m_1+m_2\geq n_1$ we do not need IA to achieve the DoF region's upper bound. Thus, the second expression of I_1 in (26) gives the dimension of the subspace spanned by the interference at the first receiver. By a similar argument as in the

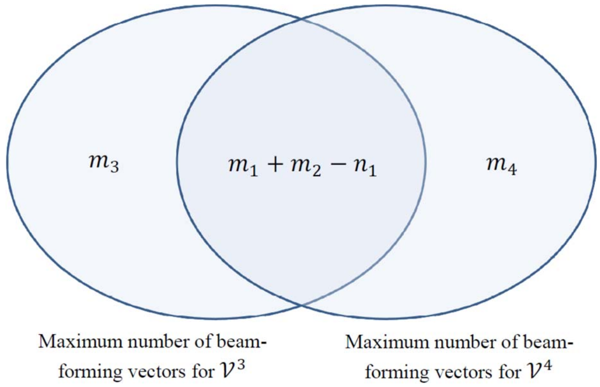

Fig. 5. Venn diagram of the number of beamforming vectors for $m_1 + m_2 > n_1$.

proof of Theorem 2, we can use linear zero-forcing receiver at receiver 1 provided that $I_1 + d_1 + d_2 \leq n_1$.

It follows that all integer-valued DoFs (d_1, d_2, d_3, d_4) that satisfy all seven bounds of $\mathcal{D}^1_{\text{in}}$ and the convex hull of all such points (by time sharing) are achievable. In part D.2 of Appendix D, we show that every integer-valued point in the outer bound $\mathcal{D}^1_{\text{out}}$ is achievable by the aforementioned scheme. This completes the proof. ∎

As shown above, the convex hull of all integer DoFs in the outer bound $\mathcal{D}^1_{\text{out}}$ in (20) is achievable. If all corner points of $\mathcal{D}^1_{\text{out}}$ assume integer values, $\mathcal{D}^1_{\text{in}}$ in (24) matches $\mathcal{D}^1_{\text{out}}$. In general, if $\mathcal{D}^1_{\text{out}}$ has a noninteger corner point, there is a gap between $\mathcal{D}^1_{\text{in}}$ of Theorem 6 and $\mathcal{D}^1_{\text{out}}$. For example, if the number of antennas in all nodes is equal to m, $(d_1 = m/2, d_2 = m/2, d_3 = 0, d_4 = m/2)$ is a corner point of $\mathcal{D}^1_{\text{out}}$. For odd m, a noninteger corner point of $\mathcal{D}^1_{\text{out}}$ is obtained, which is not directly achievable by Theorem 6. However, in the following example, we will show that in the multiletter extension of the channel the schemes in Theorem 6 can be used for achieving these noninteger DoFs.

Example 3: In the scenario of this section with $m > 1$ antennas at all nodes $(d_1 = m/2, d_2 = m/2, d_3 = 0, d_4 = m/2)$ is almost surely achievable by two-symbol extension of the constant channel.

If this channel is used twice, the received signal at receiver r can be written as

$$\overline{\boldsymbol{y^r}} = \left[\overline{\boldsymbol{H^{r1}}}\ \overline{\boldsymbol{H^{r2}}}\ \overline{\boldsymbol{H^{r4}}}\right] \times \left(\sum_{i=1}^{m} x_i^1 \begin{bmatrix} \overline{\boldsymbol{v_i^{11}}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \sum_{i=1}^{m} x_i^2 \begin{bmatrix} \mathbf{0} \\ \overline{\boldsymbol{v_i^{22}}} \\ \mathbf{0} \end{bmatrix} + \sum_{i=1}^{m} x_i^4 \begin{bmatrix} \overline{\boldsymbol{v_i^{14}}} \\ \overline{\boldsymbol{v_i^{24}}} \\ \overline{\boldsymbol{v_i^{44}}} \end{bmatrix}\right) + \overline{\boldsymbol{n^r}}, \quad r = 1, 2 \tag{32}$$

where $\overline{\boldsymbol{H^{ri}}} \in \mathcal{R}^{2m \times 2m}$ is the block diagonal channel matrix that represents the channel between the rth receiver and the ith transmitter in the two-symbol extension model

$$\overline{\boldsymbol{H^{ri}}} \triangleq \begin{bmatrix} \boldsymbol{H^{ri}} & 0 \\ 0 & \boldsymbol{H^{ri}} \end{bmatrix}. \tag{33}$$

All matrices $\boldsymbol{H^{ri}}$ are assumed to be full rank and remain constant during communication. $\overline{\boldsymbol{v_i^{jj}}} \in \mathcal{R}^{2m \times 1}$ is the beamforming vector for the ith element of the codeword for message j in the two-symbol extension model, i.e., $\boldsymbol{v_i^{jj}}(t)$ is the beamforming vector for transmitting the ith element of the codeword of message j at time $t = 1, 2$,

$$\overline{\boldsymbol{v_i^{jj}}} \triangleq \begin{bmatrix} \boldsymbol{v_i^{jj}}(1) \\ \boldsymbol{v_i^{jj}}(2) \end{bmatrix} \tag{34}$$

and $\overline{\boldsymbol{n^r}}$ is the AWGN vector in the extended channel at the rth receiver.

For $m > 1$, to design the set of beamforming vectors for each message

$$\mathcal{V}^1 = \left\{ \begin{bmatrix} \overline{\boldsymbol{v_i^{11}}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \in \mathcal{R}^{6m \times 1}, i = 1, \ldots, m \right\}$$
$$\mathcal{V}^2 = \left\{ \begin{bmatrix} \mathbf{0} \\ \overline{\boldsymbol{v_i^{22}}} \\ \mathbf{0} \end{bmatrix} \in \mathcal{R}^{6m \times 1}, i = 1, \ldots, m \right\}$$
$$\mathcal{V}^4 = \left\{ \begin{bmatrix} \overline{\boldsymbol{v_i^{14}}} \\ \overline{\boldsymbol{v_i^{24}}} \\ \overline{\boldsymbol{v_i^{44}}} \end{bmatrix} \in \mathcal{R}^{6m \times 1}, i = 1, \ldots, m \right\} \tag{35}$$

we use the scheme proposed in Theorem 6 for the two-symbol extension channel model of (32). Thus, $(\overline{d_1} = m, \overline{d_2} = m, \overline{d_3} = 0, \overline{d_4} = m)$ is achievable, and $(d_1 = m/2, d_2 = m/2, d_3 = 0, d_4 = m/2)$ is the DoF per channel use.

In a network with $m = 1$ (a SISO network), similar to Example 2, it is not possible to design beamforming vectors for zero forcing or alignment. As stated before, however, in this paper, we do not deal with the DoF of the SISO system.

The following theorem establishes the exact characterization of the spatial DoF.

Theorem 7: $\mathcal{D}^1 = \mathcal{D}^1_{\text{out}}$.

Proof: Using the concept of normalized DoF, with an argument similar to [9, Lemmas 2 and 3 and Th. 11], it follows that $\mathcal{D}^1_{\text{out}}$ is the exact characterization of the DoF region. ∎

V. Cognitive Transmitters in Each Multiple-Access Channel

For the channel model described in Section II-C, the received signals at the rth receiver in (3) can be represented in the following way:

$$\boldsymbol{y^r} = \left[\boldsymbol{H^{r1}}\ \boldsymbol{H^{r2}}\ \boldsymbol{H^{r3}}\ \boldsymbol{H^{r4}}\right] \times \left(\sum_{i=1}^{d_1} x_i^1 \begin{bmatrix} \boldsymbol{v_i^{11}} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{d_2} x_i^2 \begin{bmatrix} \boldsymbol{v_i^{12}} \\ \boldsymbol{v_i^{22}} \\ 0 \\ 0 \end{bmatrix} + \sum_{i=1}^{d_3} x_i^3 \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{v_i^{33}} \\ 0 \end{bmatrix} + \sum_{i=1}^{d_4} x_i^4 \begin{bmatrix} 0 \\ 0 \\ \boldsymbol{v_i^{34}} \\ \boldsymbol{v_i^{44}} \end{bmatrix}\right) + \boldsymbol{n^r}, \quad r = 1, 2. \tag{36}$$

In the following theorem, we establish an outer bound for the DoF region.

Theorem 8: The outer bound on the DoF region of the two-user 2-IMACs for the cognitive scenario described in Section II-C is

$$\mathcal{D}^2_{\text{out}} = \Bigg\{ (d_1,d_2,d_3,d_4) \in \mathcal{R}^{+^4} : d_1 \le m_1, d_3 \le m_3, \\ d_1+d_2 \le \min(m_1+m_2,n_1), \\ d_3+d_4 \le \min(m_3+m_4,n_2), \\ d_1+d_2+d_3 \le \max(m_3,n_1), \\ d_1+d_3+d_4 \le \max(m_1,n_2), \\ \sum_{i=1}^{4} d_i \le \max(m_1+m_2,n_2), \\ \sum_{i=1}^{4} d_i \le \max(m_3+m_4,n_1) \Bigg\}. \tag{37}$$

Proof: The first four bounds are trivial bounds on the DoF of MAC components of 2-IMAC with message sharing. The proofs for the other bounds are similar to the proof of Theorem 1. ∎

To establish the achievable DoF region for this scenario, we need the following theorem (the proof is given in Appendix C).

Theorem 9: Consider matrices $[\boldsymbol{A}]$ and $[\boldsymbol{A} \quad \boldsymbol{B}]$. $\boldsymbol{A} \in \mathcal{R}^{n\times m}$ and $\boldsymbol{B} \in \mathcal{R}^{n\times m_b}$ are random matrices with entries drawn from a continuous probability distribution independently, so both of them as well as their concatenation are full rank almost surely. If $[\boldsymbol{A}]\boldsymbol{N_1} = \boldsymbol{0}$ and $[\boldsymbol{A} \quad \boldsymbol{B}]\boldsymbol{N_2} = [\boldsymbol{A} \quad \boldsymbol{B}]\begin{bmatrix}\boldsymbol{N_{2u}}\\ \boldsymbol{N_{2d}}\end{bmatrix} = \boldsymbol{0}$, where $\boldsymbol{N_1} \in \mathcal{R}^{m\times(m-n)^+}$, $\boldsymbol{N_{2u}} \in \mathcal{R}^{m\times(m+m_b-n)^+}$, and $\boldsymbol{N_{2d}} \in \mathcal{R}^{m_b\times(m+m_b-n)^+}$, then the number of independent columns in

$$\boldsymbol{W} = \begin{bmatrix}\boldsymbol{N_1} & \boldsymbol{N_{2u}}\\ \boldsymbol{0} & \boldsymbol{N_{2d}}\end{bmatrix} \tag{38}$$

will be

$$\boldsymbol{f}_2 = (\boldsymbol{m}+\boldsymbol{m_b}-\boldsymbol{n})^+. \tag{39}$$

We now proceed to derive an achievable DoF region for the channel model in this section.

Theorem 10: The exact characterization of the DoF region of the scenario described in Section II-C is $\mathcal{D}^2 = \mathcal{D}^2_{\text{in}} = \mathcal{D}^2_{\text{out}}$, with

$$\mathcal{D}^2_{\text{in}} \triangleq convh\left(\mathcal{D}^2_{\text{out}} \cap \mathcal{Z}^{+^4}\right) \\ = convh\Bigg(\Big\{(d_1,d_2,d_3,d_4) \in \mathcal{Z}^{+^4} : \\ d_1 \le m_1, d_3 \le m_3, d_2 \le m_1+m_2 \\ d_4 \le m_3+m_4 \sum_{i=1}^{4} d_i \le \sum_{i=1}^{4} m_i \\ I_2+d_3+d_4 \le n_2, I_1+d_1+d_2 \le n_1\Big\}\Bigg) \tag{40}$$

where

$$I_1 \triangleq \begin{cases}(d_3+(d_4-m_4)^+-(m_3-n_1))^+, & \text{if } m_3>n_1\\ d_3+(d_4-(m_3+m_4-n_1)^+)^+, & \text{if } m_3\le n_1\end{cases} \tag{41}$$

$$I_2 \triangleq \begin{cases}(d_1+(d_2-m_2)^+-(m_1-n_2))^+, & \text{if } m_1>n_2\\ d_1+(d_2-(m_1+m_2-n_2)^+)^+, & \text{if } m_1\le n_2.\end{cases} \tag{42}$$

Proof: The scheme of achieving $\mathcal{D}^2_{\text{in}}$ is based on the techniques used in [16] for achieving the DoF region of the two-user IC. As in the proof of Theorem 2, based on (36), we should design

$$\mathcal{V}^1 = \left\{\begin{bmatrix}\boldsymbol{v_i^{11}}\\ \boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1},\ i=1,\dots,d_1\right\}$$

$$\mathcal{V}^2 = \left\{\begin{bmatrix}\boldsymbol{v_i^{12}}\\ \boldsymbol{v_i^{22}}\\ \boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1},\ i=1,\dots,d_2\right\}$$

$$\mathcal{V}^3 = \left\{\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{v_i^{33}}\\ \boldsymbol{0}\end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1},\ i=1,\dots,d_3\right\}$$

$$\mathcal{V}^4 = \left\{\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{v_i^{34}}\\ \boldsymbol{v_i^{44}}\end{bmatrix} \in \mathcal{R}^{\left(\sum_{i=1}^4 m_i\right)\times 1},\ i=1,\dots,d_4\right\}. \tag{43}$$

By a similar reasoning as in the proof of Theorem 2, all vectors in the union of $\bigcup_{i=1}^4 \mathcal{V}^i$ must be linearly independent. This condition is guaranteed by the first five bounds of $\mathcal{D}^2_{\text{in}}$ in (40).

At the first receiver, s_3 and s_4 are interfering messages. Using Theorem 9, we design the beamforming vectors for zero forcing the interfering signals by transmitters. In this scenario, IA is not required to achieve the upper bound. According to Theorem 9, we should consider two different cases: $m_3 \le n_1$ and $m_3 > n_1$.

In case of $m_3 \le n_1$, we can select the first $(m_3+m_4-n_1)^+$ vectors of the set $\mathcal{V}^4$ as the basis of the null space of $[\boldsymbol{H^{13}}\ \boldsymbol{H^{14}}]$. Then, we select the remaining vectors of $\mathcal{V}^4$, if $d_4 > (m_3+m_4-n_1)^+$, and all d_3 vectors in $\mathcal{V}^3$ from a continuous probability distribution at random. Thus, these $d_3 + d_4$ beamforming vectors are linearly independent almost surely, and the dimension of the subspace spanned by the interference at the first receiver is given by the second expression of I_1 in (41).

In case of $m_3 > n_1$, we have $m_3+m_4-n_1$ vectors as the basis of the null space of $[\boldsymbol{H^{13}}\ \boldsymbol{H^{14}}]$ and m_3-n_1 vectors as the basis of the null space of $\boldsymbol{H^{13}}$. However, according to Theorem 9, these two null spaces have m_3-n_1 common dimensions. The number of possible beamforming vectors for zero forcing by transmitters is demonstrated in the Venn diagram of Fig. 6.

In this case, we select the first m_4 vectors of $\mathcal{V}^4$ as the basis of the null space of $[\boldsymbol{H^{13}}\ \boldsymbol{H^{14}}]$. If $d_3>0$ or $d_4>m_4$, it is

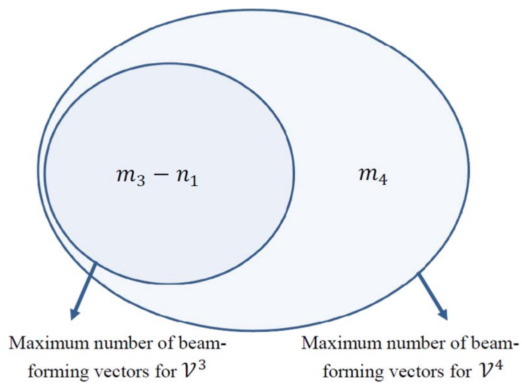

Fig. 6. Venn diagram of the number of beamforming vectors for $m_3 > n_1$.

possible to select $m_3 - n_1$ more beamforming vectors either from the null space of $\boldsymbol{H}^{\mathbf{13}}$ for $\mathcal{V}^3$ or from the null space of $[\boldsymbol{H}^{\mathbf{13}}\ \boldsymbol{H}^{\mathbf{14}}]$ for $\mathcal{V}^4$, or a combination of $m_3 - n_1$ such vectors for $\mathcal{V}^3$ and $\mathcal{V}^4$. According to Theorem 9, we cannot select more beamforming vectors for zero forcing by transmitter; otherwise, the union of $\mathcal{V}^3 \cup \mathcal{V}^4$ will no longer be a set of linearly independent vectors. If more vectors are required, we generate them at random from a continuous probability distribution. Thus, $\mathcal{V}^3 \cup \mathcal{V}^4$ contains $d_3 + d_4$ linearly independent vectors. The first expression of I_1 in (41) is the dimension of the subspace spanned by the interference at the first receiver. By a similar argument as in the proof of Theorem 2, we can use linear zero-forcing receiver at receiver 1 provided that $I_1 + d_1 + d_2 \leq n_1$. The $d_1 + d_2$ vectors of $\mathcal{V}^1 \cup \mathcal{V}^2$ can be designed in a similar way.

Therefore, all integer-valued DoFs (d_1, d_2, d_3, d_4) that satisfy all seven bounds of $\mathcal{D}^2_{\text{in}}$ and the convex hull of all such points (by time sharing) are achievable. In part D.3 of Appendix D, we will show that every integer-valued point in the outer bound $\mathcal{D}^2_{\text{out}}$ is achievable by the aforementioned scheme.

It can readily be verified that all corner points of $\mathcal{D}^2_{\text{out}}$ region are integers in the following way. $\mathcal{D}^2_{\text{out}}$ is defined by 12 bounds: 8 bounds in (37) and 4 bounds of $d_i \geq 0$, $i = 1, \ldots, 4$. A corner point of $\mathcal{D}^2_{\text{out}}$ is the solution for the nonhomogeneous system of four linear equations with four variables (d_1, d_2, d_3, d_4). By selecting 4 out of 12 bounds, we will have $\binom{12}{4} = 495$ of such systems with all entries of their coefficient matrix equal to 0 or 1. A nonhomogeneous system has a unique solution, i.e., it describes a corner point of $\mathcal{D}^2_{\text{out}}$, if its coefficient matrix is invertible. It can be verified that only 224 out of 495 systems have an invertible coefficient matrix. Moreover, the inverse of none of these 224 coefficient matrices has noninteger entries. Therefore, any possible corner points of $\mathcal{D}^2_{\text{out}}$ will not assume a noninteger value. Since $\mathcal{D}^2_{\text{out}}$ is a convex set with integer corner points, it matches the achievable region of $\mathcal{D}^2_{\text{in}}$. This yields the exact characterization of the DoF region of the channel in this section. This completes the proof. ■

VI. Discussion and Comparison of the Three Scenarios

In the last three sections, we derived the exact characterization of the DoF region of two-user 2-IMACs for three different scenarios. This section deals with the evaluation of the benefits of cognitive message sharing on increasing the transmission rates of the network.

The three DoF regions can be compared under the simplifying assumption of m antennas at all nodes

$$\mathcal{D}^1_{\forall m_i = \forall n_j = m} = \Big\{ (d_1, d_2, d_3, d_4) \in \mathcal{R}^{+^4} : d_1 + d_2 \leq m \quad d_1 + d_2 + d_3 \leq 3m, d_1 + d_2 + d_4 \leq 3m \quad d_2 + d_3 + d_4 \leq m, d_1 + d_3 + d_4 \leq m \quad \sum_{i=1}^{4} d_i \leq 2m \Big\} \tag{44}$$

$$\mathcal{D}^2_{\forall m_i = \forall n_j = m} = \Big\{ (d_1, d_2, d_3, d_4) \in \mathcal{R}^{+^4} : d_1 + d_2 + d_3 \leq m \quad d_1 + d_3 + d_4 \leq m, \sum_{i=1}^{4} d_i \leq 2m \Big\} \tag{45}$$

$$\mathcal{D}^0_{\forall m_i = \forall n_j = m} = \Big\{ (d_1, d_2, d_3, d_4) \in \mathcal{R}^{+^4} : d_1 + d_2 + d_3 \leq m \quad d_1 + d_2 + d_4 \leq m, d_2 + d_3 + d_4 \leq m \quad d_1 + d_3 + d_4 \leq m \Big\}. \tag{46}$$

It should be noted that all regions under study in this paper are 4-D, i.e., cannot be illustrated by figures. Here, we use the maximum sum DoF as a measure for studying the effect of cognition, i.e.,

$$d^{\max,i} \triangleq \max_{\mathcal{D}^i} \left(\sum_{i=1}^{4} d_i \right), \quad i = 0, 1, 2. \tag{47}$$

In the following theorem, we calculate the maximum sum DoF for three scenarios.

Theorem 11: $d^{\max,i} \triangleq \max_{\mathcal{D}^i}(\sum_{i=1}^4 d_i) = \min(\mathcal{S}^i)$, $i = 0, 1, 2$, in which we have (48)–(50), shown at the bottom of the next page.

Proof: The dual of the linear programming problem $\max_{\mathcal{D}^i}(\sum_{i=1}^4 d_i)$, $i = 0, 1, 2$, can be solved in the following way. For each scenario $(i = 0, 1, 2)$, we have calculated all possible values of $\sum_{i=1}^4 d_i$ at the boundary of $\mathcal{D}^i_{\text{out}}$ region. The number of extreme values for $\sum_{i=1}^4 d_i$ with constraints of $\mathcal{D}^0_{\text{out}}$, $\mathcal{D}^1_{\text{out}}$, and $\mathcal{D}^2_{\text{out}}$ is 207, 63, and 7, respectively. If we omit the redundant values for the problem $d^{\max,i}_{\text{out}}$, according to the fundamental theorem of linear programming, the minimum of 11 values in $\mathcal{S}^0$, 6 values in $\mathcal{S}^1$, and 3 values in $\mathcal{S}^2$ determines $d^{\max,i}_{\text{out}}$ for $i = 0, 1, 2$, respectively. Redundant values for $d^{\max,i}_{\text{out}}$ are those values that can be written as a linear combination of values in $\mathcal{S}^i$, $i = 0, 1, 2$. Since the three outer bounds are achievable, we have $d^{\max,i} = d^{\max,i}_{\text{out}}$. ■

An immediate result of the above theorem is that, for a network with m antennas at all nodes, if there is no cognition, the maximum sum DoF increases by $4m/3$. However, for the first and second cognitive scenarios, it increases by $3m/2$ and $2m$, respectively. This observation suggests that for a cognitive cellular network, with the same number of antennas at all nodes, allowing the CUs to exist in each cell (as defined in Section II-C) is more rate efficient than a cognitive cell in close proximity of a primary cell (as defined in Section II-B).

We can also use the above theorem to determine when cognition will be beneficial in terms of maximum sum DoF. It is obvious that cognition does not decrease the DoF. However, it is reasonable to investigate when it can strictly increase the total DoF in a network.

Corollary 1: For the message sharing scenario of Section II-C, $d^{\max,0} < d^{\max,2}$ iff at least one of the seven elements of $\mathcal{S}^0 - \mathcal{S}$ is less than all the three elements of $\mathcal{S}^2$, where

$$\mathcal{S} = \left\{ \sum_{i=1}^{4} m_i,\ n_1 + n_2,\ \max(m_1 + m_2, n_2),\ \max(m_3 + m_4, n_1) \right\}. \quad (51)$$

Proof: We have $\min(m_1 + m_2, n_2) + \min(m_3 + m_4, n_1) \leq \min(\sum m_i, n_1 + n_2)$. Since the outer bounds of $\mathcal{D}^0_{\text{out}}$ and $\mathcal{D}^2_{\text{out}}$ are achievable, if at least one element of $\mathcal{S}^0 - \mathcal{S}$ is less than all the elements of $\mathcal{S}^2$, we have $d^{\max,0} < d^{\max,2}$. Conversely, if $d^{\max,0} < d^{\max,2}$, then at least one element of $\mathcal{S}^0 - \mathcal{S}$ will be less than all the elements of $\mathcal{S}^2$. ■

Corollary 2: For the message sharing scenario of Section II-B, $d^{\max,0} < d^{\max,1}$ iff at least one of the five elements of $\mathcal{S}^0 - \mathcal{S}^1$ is less than all the six elements of $\mathcal{S}^1$. ■

Proof: Since $\mathcal{S}^1 \subset \mathcal{S}^0$, the result follows. ■

It is also notable that the maximum sum DoF of $d^{\max,2}$ is equal to the maximum sum DoF of the corresponding two-user MIMO IC [4]. If $d_1 = d_3 = 0$ or if the two transmitters of each MAC share their messages, the system will be reduced to a two-user IC. However, the DoF region of $\mathcal{D}^2$ is different from the DoF region of the two-user IC in [16]. This can be observed, for example, by setting $d_1 = d_3 = 0$ in (45), which yields the DoF region of the corresponding two-user IC, i.e.,

$$\mathcal{D}^2_{\substack{\forall m_i = \forall n_j = m \\ d_1 = d_3 = 0}} = \left\{ (d_2, d_4) \in \mathcal{R}^{+^2} : d_2 \leq m,\ d_4 \leq m \right\}. \quad (52)$$

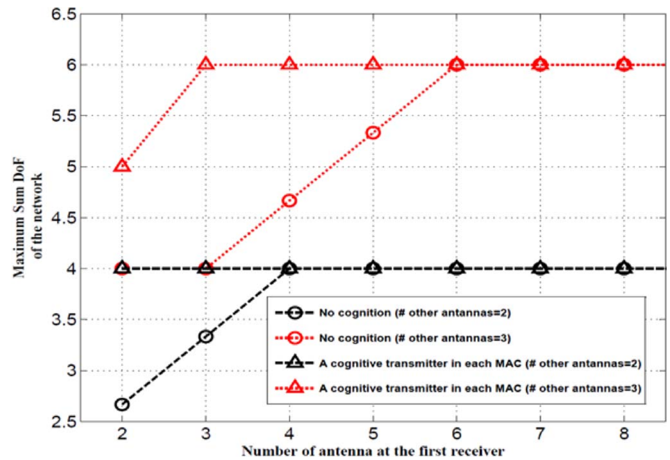

Fig. 7. Maximum sum DoF of the second cognitive scenario in comparison with no-cognition as a function of n_1 number of antennas at other nodes is equal to 2 and 3.

It is also interesting to compare the total DoF of two cognitive scenarios with the no-cognition scenario, as the number of antennas changes.

In Fig. 7, we have studied $d^{\max,0}$ and $d^{\max,2}$ as functions of the number of antennas at the first receiver, with m antennas at all the other nodes (for $m = 2$ and $m = 3$). For $m = 2$, the maximum possible sum DoF is 4, and for $m = 3$, it is 6. It can be observed that, in these settings, when there is a CU in each MAC the maximum possible sum DoF can be achieved with fewer antennas at the receiver. For $m = 2$, we need four antennas at the first receiver to achieve the maximum possible total DoF of four, whereas in a cognitive network, two antennas at the receiver are sufficient for achieving four DoFs. For $m = 3$, the first receiver must be equipped with at least six antennas to achieve the maximum total DoF of the system, whereas in a cognitive network, we can achieve the maximum sum DoF with only three antennas at the receiver.

We have also studied $d^{\max,0}$ and $d^{\max,1}$ as functions of the number of antennas in the first receiver (cognitive MAC's

$$\mathcal{S}^0 = \left\{ \sum_{i=1}^{4} m_i, n_1 + n_2, m_1 + \max(m_2, n_2),\ m_2 + \max(m_1, n_2),\ m_3 + \max(m_4, n_1),\ m_4 + \max(m_3, n_1), \max(m_3 + m_4, n_1),\ \max(m_1 + m_2, n_2),\ \frac{n_2 + \max(m_4, n_1) + \max(m_3, n_1)}{2}, \frac{n_1 + \max(m_2, n_2) + \max(m_1, n_2)}{2}, \frac{\max(m_4, n_1) + \max(m_3, n_1) + \max(m_2, n_2) + \max(m_1, n_2)}{3} \right\} \quad (48)$$

$$\mathcal{S}^1 = \left\{ \sum_{i=1}^{4} m_i, n_1 + n_2,\ m_1 + \max(m_2, n_2),\ m_2 + \max(m_1, n_2),\ \max(m_1 + m_2, n_2), \frac{n_1 + \max(m_2, n_2) + \max(m_1, n_2)}{2} \right\} \quad (49)$$

$$\mathcal{S}^2 = \{\min(m_1 + m_2, n_1) + \min(m_3 + m_4, n_2),\ \max(m_1 + m_2, n_2),\ \max(m_3 + m_4, n_1)\} \quad (50)$$

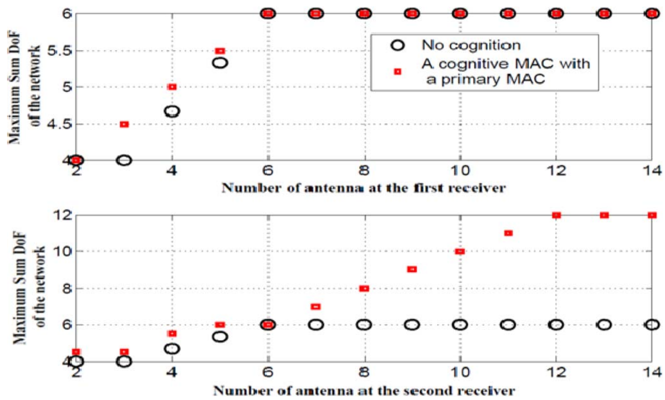

Fig. 8. Maximum sum DoF of the first cognitive scenario in comparison with no-cognition as a function of n_1 (up) and n_2 (down) number of antennas at other nodes is equal to 3.

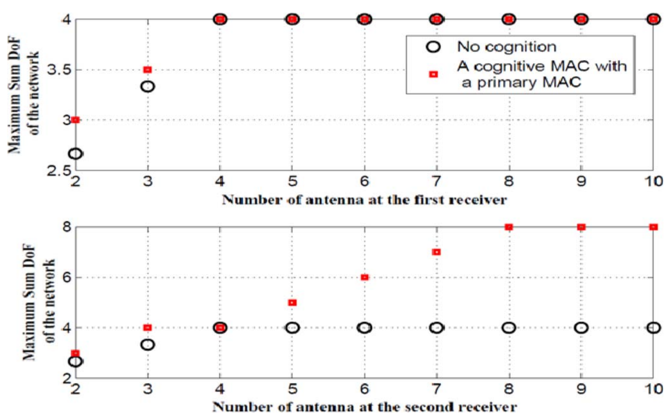

Fig. 9. Maximum sum DoF of the first cognitive scenario in comparison with no-cognition as a function of n_1 (up) and n_2 (down) number of antennas at other nodes is equal to 2.

receiver) and the second receiver (primary MAC's receiver). For each of these two cases, the number of antennas at other nodes is $m = 2$, as in Fig. 9, and $m = 3$, as in Fig. 8.

From the upper half of Figs. 8 and 9, we observe that the total possible DoF of the no-cognition scenario is close to the total DoF of a cognitive MAC in close proximity of the primary MAC scenario, when the number of antennas at the cognitive MAC's receiver changes and the number of antennas at other nodes remains constant. However, according to the lower half of Figs. 8 and 9, increasing the number of antennas at the primary MAC's receiver (second receiver) will increase the maximum possible sum DoF of the network. Therefore, in such settings of the cognitive network, increasing the number of antennas of the primary MAC's receiver is more rate efficient than increasing the number of antennas of the cognitive MAC's receiver.

VII. Conclusion

In this paper, we have studied the optimal spatial DoF of a network by performing zero-forcing interference by transmitters and IA in a network with more than two transmitters and genie-aided cognition. Using some techniques from matrix theory, we formulated the maximum number of linearly independent beamforming vectors for zero-forcing interference by transmitters. Then, we established the exact characterization of the DoF region of the two-user 2-IMACs for no message sharing and two cognitive message sharing scenarios. For the no message sharing scenario, using the algorithm introduced in Appendix B, the achievability is generalized into a network with more than two transmitters in each MAC.

We observed that IA is not required for two-user 2-IMACs when a cognitive transmitter exists in each MAC to achieve the optimal DoF region. However, in general, we require IA to achieve the optimal DoFs for no message sharing and for a cognitive MAC in close proximity of a primary MAC. For these two scenarios, if the outer bound region assumes noninteger corner points, there will be a gap between the achievable region and the outer bound, which can be resolved by applying the proposed achievability scheme for the multiletter extension of the channel.

Insightful results were also provided by comparing optimal DoFs in terms of the number of antennas in different nodes and different message sharing scenarios.

Appendix A
Alignment Lemma

Lemma A.1: Let $\boldsymbol{H^k} \in \mathcal{R}^{n\times m_k}$, $k = 1, \ldots, K$, be random matrices from K interfering transmitters to a common receiver. If their entries are drawn from a continuous probability distribution independently (they are full rank almost surely), the dimension of the intersection subspace between their column spaces is almost surely equal to

$$C = \left(\sum_{k=1}^{K} m_k - (K-1)n - \sum_{k=1}^{K}(m_k - n)^+\right)^+. \quad \text{(A.1)}$$

Proof: Let

$$\boldsymbol{H} \triangleq \begin{bmatrix} \boldsymbol{I_n} & -\boldsymbol{H^1} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{I_n} & \boldsymbol{0} & \cdots & -\boldsymbol{H^K} \end{bmatrix} \quad \text{(A.2)}$$

$$\boldsymbol{HN} = \begin{bmatrix} \boldsymbol{I_n} & -\boldsymbol{H^1} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{I_n} & \boldsymbol{0} & \cdots & -\boldsymbol{H^K} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & & \boldsymbol{0} & \boldsymbol{q^1} & & \boldsymbol{q^C} \\ \boldsymbol{N}_1 & \boldsymbol{0} & & \boldsymbol{0} & \boldsymbol{q_1^1} & & \boldsymbol{q_1^C} \\ \boldsymbol{0} & \boldsymbol{N_2} & & \vdots & \boldsymbol{q_2^1} & & \boldsymbol{q_2^C} \\ \vdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \vdots & \cdots & \vdots \\ \boldsymbol{0} & \vdots & & \boldsymbol{0} & \boldsymbol{q_{K-1}^1} & & \boldsymbol{q_{K-1}^C} \\ \boldsymbol{0} & \boldsymbol{0} & & \boldsymbol{N_K} & \boldsymbol{q_K^1} & & \boldsymbol{q_K^C} \end{bmatrix}_{C\,\text{columns}} = \boldsymbol{0} \quad \text{(A.3)}$$

where $\boldsymbol{N}$ represents the null space of $\boldsymbol{H}$, $\boldsymbol{I_n}$ is the $n \times n$ identity matrix, and $\boldsymbol{N_k} \in \mathcal{R}^{m_k\times(m_k-n)^+}$ represents the null space of the random matrix $\boldsymbol{H^k}$ for $k = 1, \ldots, K$. If we select columns of $\boldsymbol{N_k}$ as the beamforming vectors for the kth transmitter, the interfering signal of the kth transmitter will be zero forced. Therefore, we can use $\sum_{k=1}^{K}(m_k - n)^+$ out of $(\sum_{k=1}^{K} m_k - (K-1)n)^+$ columns of $\boldsymbol{N}$ for zero-forcing interferences by transmitters.

We then use the remaining vectors of $\boldsymbol{N}$ for IA. If we select $\boldsymbol{q_k^c} \in \mathcal{R}^{m_k \times 1}$, $c = 1, \dots, C$, vectors as the beamforming vector for the kth transmitter, we have

$$\boldsymbol{q^c} = \boldsymbol{H^1 q_k^c} = \cdots = \boldsymbol{H^K q_K^c}, \quad \boldsymbol{c} = 1, \dots, \boldsymbol{C} \tag{A.4}$$

and the K interfering signals are aligned along one dimension at the receiver, where almost surely

$$\begin{aligned} C &= \left(\left(\sum_{k=1}^{K} m_k - (K-1)n \right)^+ - \sum_{k=1}^{K} (m_k - n)^+ \right)^+ \\ &= \left(\sum_{k=1}^{K} m_k - (K-1)n - \sum_{k=1}^{K} (m_k - n)^+ \right)^+ . \end{aligned} \tag{A.5}$$

This is the dimension of the intersection subspace between the column spaces of these channel matrices. ■

APPENDIX B
ALIGNMENT ALGORITHM

Consider a part of a network in which K transmitters cause interference to one receiver. Therefore, matrices $\boldsymbol{H^k} \in \mathcal{R}^{n \times m_k}$, $k = 1, \dots, K$, describe the interfering links. Our aim is to design the set of beamforming vectors

$$\mathcal{V}^k = \left\{ \boldsymbol{v_i^k} \in \mathcal{R}^{\boldsymbol{m_k} \times 1},\ i = 1, \dots, d_k \right\}$$

in a way that they occupy a few dimensions in the n-dimensional signal space of this receiver. For zero forcing by transmitter, we select the first $(m_k - n)^+$ vectors of $\mathcal{V}^k$ as the basis of the null space of $\boldsymbol{H^k}$. We continue to design the remaining vectors of each set $\mathcal{V}^k$ to be aligned at the receiver by the following algorithm:

Step 0) set $d'_k \leftarrow (d_k - (m_k - n)^+)^+$ for $k \in \{1, \dots, K\}$.

Step 1) Using Lemma A.1, calculate the dimension of the intersection subspace between the column spaces of all matrices $\boldsymbol{H^k}$, $k = 1, \dots, K$, and place it in $C_{\{1,\dots,K\}}$. In the same way, calculate $C_{\mathcal{X}}$ in which $\mathcal{X}$ is any subset of $\{1, \dots, K\}$. (For example, $C_{\{1,2,3\}}$ is the dimension of the intersection subspace between column spaces of $\boldsymbol{H^k}$, $k = 1, 2, 3$.)

Step 2) Alignment of the interference from K transmitters:

- 2-a) design $r_K \triangleq \min\{C_{\{1,\dots,K\}}, d'_k : k \in \{1, \dots, K\}\}$ vectors of sets $\mathcal{V}^k$, $k \in \{1, \dots, K\}$ to be aligned at the receiver.
- 2-b) set $d'_k \leftarrow d'_k - r_K$, $k \in \{1, \dots, K\}$ and $C_{\mathcal{X}} \leftarrow C_{\mathcal{X}} - r_K$, $\forall \mathcal{X} \subset \{1, \dots, K\}$.

Step 3) set $I \leftarrow K - 1$.

Step 4) alignment of the interference from I transmitters:

- 4-a) let $r_I \triangleq \max_{\forall \mathcal{X}: \|\mathcal{X}\| = I} (\min\{C_{\mathcal{X}}, d'_k : k \in \mathcal{X}\})$. If $\mathcal{X}_0$ maximizes r_I, then design the remaining r_I vectors of $\mathcal{V}^k$, $k \in \mathcal{X}_0$ so that they are aligned at the receiver.
- 4-b) set $d'_k \leftarrow d'_k - r_I$, $k \in \mathcal{X}_0$ and $C_{\mathcal{X}} \leftarrow C_{\mathcal{X}} - r_I$, $\forall \mathcal{X} \subset \mathcal{X}_0$.
- 4-c) repeat steps 4-a and 4-b until $r_I = 0$.
- 4-d) set $I \leftarrow I - 1$.

Step 5) if $I \geq 2$ repeat step 4.

Step 6) End.

APPENDIX C
PROOF OF THEOREMS 5 AND 9

Lemma C.1: Consider matrix $[\boldsymbol{A} \quad \boldsymbol{B}]$. $\boldsymbol{A} \in \mathcal{R}^{n \times m_a}$ and $\boldsymbol{B} \in \mathcal{R}^{n \times m_b}$ are random matrices that their entries are drawn independently from a continuous probability distribution, so that both of them as well as their concatenation are full rank almost surely. If $m_a + m_b > n$ and $[\boldsymbol{A} \quad \boldsymbol{B}]\boldsymbol{N} = [\boldsymbol{A} \quad \boldsymbol{B}] \begin{bmatrix} \boldsymbol{N_u} \\ \boldsymbol{N_d} \end{bmatrix} = \boldsymbol{0}$, then both submatrices of $\boldsymbol{N}$, i.e., $\boldsymbol{N_u} \in \mathcal{R}^{m_a \times (m_a + m_b - n)}$ and $\boldsymbol{N_d} \in \mathcal{R}^{m_b \times (m_a + m_b - n)}$, are almost surely full rank.

Proof: We use Sylvester rank bounds; for any two matrices $\boldsymbol{X} \in \mathcal{R}^{p \times q}$ and $\boldsymbol{X} \in \mathcal{R}^{q \times r}$ with $rank(\boldsymbol{X}) = r_x$ and $rank(\boldsymbol{Y}) = r_y$ we have [20]

$$r_x + r_y - q \leq rank(\boldsymbol{XY}) \leq \min(r_x, r_y). \tag{C.1}$$

Here, it is assumed that $rank(\boldsymbol{A}) = min(m_a, n)$ and $rank(\boldsymbol{B}) = min(m_b, n)$. Furthermore, by definition, $rank(\boldsymbol{N}) = m_a + m_b - n$. We prove the lemma for two different cases.

First case: $m_a > n$ and $m_b > n$, i.e., both submatrices are broad (the number of columns are more than the number of rows). We can write

$$\begin{aligned} &[\boldsymbol{A} \quad \boldsymbol{B}] \begin{bmatrix} \boldsymbol{N_u} \\ \boldsymbol{N_d} \end{bmatrix} = \boldsymbol{0} \\ &\Rightarrow \quad \boldsymbol{A N_u} = -\boldsymbol{B N_d} = \boldsymbol{X} \in \mathcal{R}^{n \times (m_a + m_b - n)}. \end{aligned} \tag{C.2}$$

We assume $rank(\boldsymbol{N_u}) = m_a - x$ and $rank(\boldsymbol{N_d}) = m_b - y$, and the row space of $\boldsymbol{N_u}$ and $\boldsymbol{N_d}$ has a k-dimensional common subspace (therefore, $rank(\boldsymbol{X}) \leq k$). We show that $x = y = 0$.

Since $rank(\boldsymbol{N}) = m_a + m_b - n$, we have

$$m_a - y + m_b - x - k = m_a + m_b - n \Rightarrow x + y + k = n. \tag{C.3}$$

By the Sylvester lower bound

$$\begin{aligned} &n + (m_a - x) - m_a \leq rank(\boldsymbol{X} = \boldsymbol{A N}_u) \leq k \\ &\overset{x+y+k=n}{\Rightarrow} y + k \leq k \Rightarrow y = 0 \end{aligned} \tag{C.4}$$

and in the same way

$$\begin{aligned} &n + (m_b - y) - m_b \leq rank(\boldsymbol{X} = -\boldsymbol{B N}_d) \leq k \\ &\overset{x+y+k=n}{\Rightarrow} x + k \leq k \Rightarrow x = 0. \end{aligned} \tag{C.5}$$

Second case: at least one of $\boldsymbol{N_u}$ or $\boldsymbol{N_d}$ is not broad. We show by contradiction that the tall (or square) submatrix is full rank. Then, using Sylvester bounds, we can show that the broad submatrix is also full rank. Here, we present this scheme for $\boldsymbol{N_u}$ tall (or square) and $\boldsymbol{N_d}$ broad, i.e., for $m_a > n$ and $m_b \leq n$.

If we assume $\boldsymbol{N_u}$ is not a full rank matrix, we can write

$$rank(\boldsymbol{N_u}) \leq m_a + m_b - n$$
$$\Rightarrow \exists \boldsymbol{x_u} \in \mathcal{R}^{m_a+m_b-n}, \boldsymbol{x_u} \neq 0 : \boldsymbol{N_u x_u} = \boldsymbol{0}. \quad \text{(C.6)}$$

From $\boldsymbol{AN_u} = -\boldsymbol{BN_d}$, we will have the system of $\boldsymbol{BN_d x_u} = \boldsymbol{0}$. Since $rank(\boldsymbol{B}) = \min(m_b, n) = m_b$, the only solution to this system is the trivial solution of $\boldsymbol{N_d x_u} = \boldsymbol{0}$. Therefore, $\begin{bmatrix} \boldsymbol{N_u} \\ \boldsymbol{N_d} \end{bmatrix} \boldsymbol{x_u} = \boldsymbol{0}, \ \boldsymbol{x_u} \neq \boldsymbol{0}$. However, $rank(\boldsymbol{N}) = m_a + m_b - n$ yields $\boldsymbol{x_u} = \boldsymbol{0}$, and this is a contradiction. Thus, $rank(\boldsymbol{N_u}) = m_a + m_b - n$. This method can always be used to show that the tall (or square) submatrix is full rank.

Then, we show that the broad submatrix is also full rank. If $rank(\boldsymbol{N_d}) = r$, we have

$$\begin{aligned} m_b + (r) - m_b &\leq r_x \\ &= rank(\boldsymbol{X} = -\boldsymbol{BN_d}) \leq \min(r, m_b) \\ &\Rightarrow r_x = r \leq m_b \end{aligned} \quad \text{(C.7)}$$
$$\begin{aligned} n + (m_a + m_b - n) - m_a &\leq r_x \\ &= rank(\boldsymbol{X} = \boldsymbol{AN_u}) \\ &\Rightarrow r_x \geq m_b. \end{aligned} \quad \text{(C.8)}$$

Thus, $r_x = r = m_b$. Moreover, $\boldsymbol{N_d}$ is also full rank.

If one of $\boldsymbol{N_u}$ or $\boldsymbol{N_d}$ is tall (or square), by contradiction we can show that the tall (or square) matrix is full rank. Then, using Sylvester bounds, we can show that the broad submatrix is also full rank. If $\boldsymbol{N_u}$ and $\boldsymbol{N_d}$ are tall (or square), the contradiction suffices to show that both of them are full rank. ■

Proof of Theorem 5: By definition, both $\begin{bmatrix} \boldsymbol{N_{1u}} \\ \boldsymbol{N_{1d}} \\ \boldsymbol{0} \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{N_{2u}} \\ \boldsymbol{0} \\ \boldsymbol{N_{2d}} \end{bmatrix}$ are full rank matrices. Considering two different cases, we will prove that the dimension of the intersection subspace between their column spaces is $(m-n)^+$.

The first case arises when $m \leq n$, $m + m_1 - n > 0$, and $m + m_2 - n > 0$. In this case, both $\boldsymbol{N_1}$ and $\boldsymbol{N_2}$ are tall and full rank matrices by Lemma C.1. Therefore, any linear combination of columns of $\boldsymbol{N_{1d}}$ and $\boldsymbol{N_{2d}}$ with nonzero coefficients cannot be equal to zero, and W is full rank with $2m + m_1 + m_2 - 2n$ linearly independent columns.

The second case arises when $m > n$. In this case, both $\boldsymbol{N_{1d}}$ and $\boldsymbol{N_{2d}}$ are broad matrices. If we select all columns of $\begin{bmatrix} \boldsymbol{N_{1u}} \\ \boldsymbol{N_{1d}} \\ \boldsymbol{0} \end{bmatrix}$ and m_2 columns of $\begin{bmatrix} \boldsymbol{N_{2u}} \\ \boldsymbol{0} \\ \boldsymbol{N_{2d}} \end{bmatrix}$ corresponding to m_2 linearly independent columns of $\boldsymbol{N_{2d}}$ (note that by Lemma C.1 there are always such m_2 columns), all such $m + m_1 - n + m_2$ columns are linearly independent. Now, we show that any other column of $\begin{bmatrix} \boldsymbol{N_{2u}} \\ \boldsymbol{0} \\ \boldsymbol{N_{2d}} \end{bmatrix}$ lies in the span of these $m + m_1 - n + m_2$ selected columns. In other words, $W' = \begin{bmatrix} \boldsymbol{N_{1u}} & \boldsymbol{N'_{2u}} \\ \boldsymbol{N_{1d}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{N'_{2d}} \end{bmatrix}$ with $\begin{bmatrix} \boldsymbol{N'_{2u}} \\ \boldsymbol{0} \\ \boldsymbol{N'_{2d}} \end{bmatrix} \in \mathcal{R}^{(m+m_1+m_2)\times(m_2+1)}$ is rank deficient.

By Lemma C.1, $rank(\boldsymbol{N_{1d}}) = m_1$, and

$$\exists \boldsymbol{a} \in \mathcal{R}^{(m+m_1-n)\times 1}, \ \boldsymbol{a} \neq \boldsymbol{0} : \boldsymbol{N_{1d} a} = \boldsymbol{0}. \quad \text{(C.9)}$$

In the system of (C.9), the number of variables is more than the number of linearly independent equations ($m + m_1 - n$ variables and m_1 linearly independent equations). Therefore, the system has infinite solutions. Defining the first $m - n$ variables as free variables and $\boldsymbol{h_i}$'s as particular solutions of the system, the general solution [20] of the system can be written as

$$\boldsymbol{a} = \boldsymbol{a}_{f1}\boldsymbol{h}_1 + \cdots + a_{f(m-n)}\boldsymbol{h}_{(m-n)}. \quad \text{(C.10)}$$

By the same argument, since $rank(\boldsymbol{N'_{2d}}) = m_2$

$$\exists \boldsymbol{b} \in \mathcal{R}^{(m_2+1)\times 1}, \ \boldsymbol{b} \neq \boldsymbol{0} : \boldsymbol{N'_{2d} b} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{b} = \boldsymbol{b}_{f1}\boldsymbol{h}'_1. \quad \text{(C.11)}$$

Now, we have

$$\boldsymbol{AN_{1u}} = -\boldsymbol{B}^1\boldsymbol{N_{1d}} \Rightarrow \boldsymbol{AN_{1u}a} = \boldsymbol{0} \quad \text{(C.12)}$$
$$\boldsymbol{AN'_{2u}} = -\boldsymbol{B}^2\boldsymbol{N'_{2d}} \Rightarrow \boldsymbol{AN'_{2u}b} = \boldsymbol{0} \quad \text{(C.13)}$$

with

$$\begin{aligned} \boldsymbol{a} &= a_{f1}\boldsymbol{h}_1 + \cdots + a_{f(m-n)}\boldsymbol{h}_{(m-n)} \Rightarrow \\ \boldsymbol{N}_{1u}\boldsymbol{a} &= a_{f1}\boldsymbol{I}_1 + \cdots + a_{f(m-n)}\boldsymbol{I}_{(m-n)} \end{aligned} \quad \text{(C.14)}$$
$$\boldsymbol{b} = b_{f1}\boldsymbol{h}'_1 \Rightarrow \boldsymbol{N}'_{2d}\boldsymbol{b} = b_{f1}\boldsymbol{I}'_1. \quad \text{(C.15)}$$

Vectors $\boldsymbol{N_{1u}a}$ and $\boldsymbol{N'_{2d}b}$ are the solutions of the systems of (C.12) and (C.13). Since $rank(\boldsymbol{A}) = n$, we have exactly $m - n$ particular solutions for a system of linear equations with $\boldsymbol{A} \in \mathcal{R}^{n\times m}$ as its coefficient matrix. These particular solutions are $\boldsymbol{I}_i : i = 1, \ldots, m-n$, and $\boldsymbol{I}'_1$ can be written as a linear combination of $\boldsymbol{I}_i$'s. Therefore, the system

$$\boldsymbol{N_{1u}a} = \boldsymbol{N'_{2u}b} \quad \text{(C.16)}$$

always has a solution, and we have

$$\begin{bmatrix} \boldsymbol{N_{1u}} \\ \boldsymbol{N_{1d}} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{a} = \begin{bmatrix} \boldsymbol{N'_{2u}} \\ \boldsymbol{0} \\ \boldsymbol{N'_{2d}} \end{bmatrix} \boldsymbol{b} \Rightarrow \begin{bmatrix} \boldsymbol{N_{1u}} & \boldsymbol{N'_{2u}} \\ \boldsymbol{N_{1d}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{N'_{2d}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ -\boldsymbol{b} \end{bmatrix} = \boldsymbol{0}. \quad \text{(C.17)}$$

This completes the proof. ■

Proof of Theorem 9: If we set $m_2 = 0$ and $m_1 = m_b$ in Theorem 5, we obtain the result. ■

Appendix D
All Integer Degrees of Freedom in $\mathcal{D}^i_{\text{out}}, i = 0, 1, 2$ Are Achievable

Part D.1: We show that every integer DoF point in $\mathcal{D}^0_{\text{out}}$ is achievable by Theorem 2. I_2 in bound $I_2 + d_3 + d_4 \leq n_2$ of $\mathcal{D}^0_{\text{in}}$ can assume five values, as presented in Table IV. Here, we

TABLE IV
POSSIBLE VALUES FOR $\boldsymbol{I}_2$ IN THEOREM 2

Case	$I_2 =$
A	0
B	$d_2-(m_2-n_2)^+$
C	$d_1-(m_1-n_2)^+$
D	$d_1+d_2-(m_1+m_2-n_2)$
E	$d_1+d_2-(m_1-n_2)^+-(m_2-n_2)^+$

consider five cases for this bound. $I_1 + d_1 + d_2 \leq n_1$ can be treated in a similar way.

A) $I_2 = 0$, we have

$$I_2 + d_3 + d_4 \leq n_2 \Leftrightarrow d_3 + d_4 \leq n_2. \quad \text{(D.1)}$$

B) $I_2 = d_2 - (m_2 - n_2)^+$, we have

$$\begin{aligned} I_2 + d_3 + d_4 \leq n_2 &\Leftrightarrow d_2 - (m_2 - n_2)^+ + d_3 + d_4 \leq n_2 \\ &\Leftrightarrow d_2 + d_3 + d_4 \leq \max(m_2, n_2). \quad \text{(D.2)} \end{aligned}$$

C) $I_2 = d_1 - (m_1 - n_2)^+$, we have

$$\begin{aligned} I_2 + d_3 + d_4 \leq n_2 &\Leftrightarrow d_1 - (m_1 - n_2)^+ + d_3 + d_4 \leq n_2 \\ &\Leftrightarrow d_1 + d_3 + d_4 \leq \max(m_1, n_2). \quad \text{(D.3)} \end{aligned}$$

D) $I_2 = d_1 + d_2 - (m_1 + m_2 - n_2)$, which occurs if $m_1 + m_2 - n_2 - (m_1 - n_2)^+ - (m_2 - n_2)^+ > 0$ is the output of min in (12), we have

$$\begin{aligned} I_2 + d_3 + d_4 \leq n_2 &\Leftrightarrow \sum_{i=1}^{4} d_i \leq m_1 + m_2 \\ &\overset{m_1+m_2-n_2-(m_1-n_2)^+-(m_2-n_2)^+>0}{\Leftrightarrow} \sum d_i \\ &\leq \max(m_1 + m_2, n_2). \end{aligned} \quad \text{(D.4)}$$

E) Finally, in the case of $I_2 = d_1 + d_2 - (m_1 - n_2)^+ - (m_2 - n_2)^+$, which occurs if $m_1 + m_2 - n_2 - (m_1 - n_2)^+ - (m_2 - n_2)^+ \leq 0$, $d_1 - (m_1 - n_2)^+ > 0$, and $d_2 - (m_2 - n_2)^+ > 0$, we have

$$\begin{aligned} I_2 + d_3 + d_4 \leq n_2 &\Leftrightarrow d_1 + d_2 - (m_1 - n_2)^+ \\ &\quad - (m_2 - n_2)^+ + d_3 + d_4 \\ &\leq n_2 \Leftrightarrow \sum_{i=1}^{4} d_i \\ &\leq n_2 + (m_1 - n_2)^+ + (m_2 - n_2)^+ \\ &\overset{m_1+m_2-n_2-(m_1-n_2)^+-(m_2-n_2)^+\leq 0}{\Leftrightarrow} \sum d_i \\ &\leq \max(m_1 + m_2, n_2). \quad \text{(D.5)} \end{aligned}$$

Note that $m_1 + m_2 - n_2 - (m_1 - n_2)^+ - (m_2 - n_2)^+ \leq 0$ yields $n_2 + (m_1 - n_2)^+ + (m_2 - n_2)^+ = \max(m_1 + m_2, n_2)$.

TABLE V
POSSIBLE VALUES FOR $\boldsymbol{I}_1$ IN THEOREM 6

Case	$I_1 =$
A	0
B	$d_3-(m_1+m_2+m_3-n_1)^+$
C	$d_4-(m_1+m_2+m_4-n_1)^+$
D	$d_3+d_4-(\sum_{i=1}^{4} m_i-n_1)$
E	$d_3+d_4-(m_1+m_2+m_3-n_1)^+$ $-(m_1+m_2+m_4-n_1)^+$

Part D.2: Here, we show that every integer-valued point in the outer bound $\mathcal{D}_{\text{out}}^1$ is achievable by Theorem 6. Considering all possible values of $\boldsymbol{I}_1$ in $\mathcal{D}_{\text{in}}^1$, we should study five different cases, as presented in Table V. Possible values of $\boldsymbol{I}_2$ are considered in Part D.1.

A) $I_1 = 0$, we have

$$I_1 + d_1 + d_2 \leq n_1 \Leftrightarrow d_1 + d_2 \leq n_1. \quad \text{(D.6)}$$

B) $I_1 = d_3 - (m_1 + m_2 + m_3 - n_1)^+$, we have

$$\begin{aligned} I_1 + d_1 + d_2 \leq n_1 &\Leftrightarrow d_3 - (m_1 + m_2 + m_3 - n_1)^+ + d_1 + d_2 \\ &\leq n_1 \Leftrightarrow d_1 + d_2 + d_3 \\ &\leq \max(m_1 + m_2 + m_3, n_1). \quad \text{(D.7)} \end{aligned}$$

C) $I_1 = d_4 - (m_1 + m_2 + m_4 - n_1)^+$, we have

$$\begin{aligned} I_1 + d_1 + d_2 \leq n_1 &\Leftrightarrow d_4 - (m_1 + m_2 + m_4 - n_1)^+ + d_1 + d_2 \\ &\leq n_1 \Leftrightarrow d_1 + d_2 + d_4 \\ &\leq \max(m_1 + m_2 + m_4, n_1). \quad \text{(D.8)} \end{aligned}$$

D) $I_1 = d_3 + d_4 - (\sum_{i=1}^{4} m_i - n_1)$, we have

$$I_1 + d_1 + d_2 \leq n_1 \Leftrightarrow \sum d_i \leq \sum m_i. \quad \text{(D.9)}$$

E) Finally, in the case of $I_1 = d_3 + d_4 - (m_1 + m_2 + m_3 - n_1)^+ - (m_1 + m_2 + m_4 - n_1)^+$, which occurs if $A = 0$, $X_3 > 0$, and $X_4 > 0$, we have

$$\begin{aligned} I_1 + d_1 + d_2 \leq n_1 &\Leftrightarrow \sum d_i \leq n_1 + (m_1 + m_2 + m_3 - n_1)^+ \\ &\qquad + (m_1 + m_2 + m_4 - n_1)^+ \\ &\overset{A=0}{\Leftrightarrow} \sum d_i \leq \sum m_i. \quad \text{(D.10)} \end{aligned}$$

Part D.3: Now, we show that every integer-valued point in the outer bound $\mathcal{D}_{\text{out}}^2$ is achievable by Theorem 10. Considering all possible values of $\boldsymbol{I}_1$ in $\mathcal{D}_{\text{in}}^2$, we study five different cases, as presented in Table VI. Possible values of $\boldsymbol{I}_2$ in $\mathcal{D}_{\text{in}}^2$ can be studied in a similar way.

A) $I_1 = 0$, we have

$$I_1 + d_1 + d_2 \leq n_1 \Leftrightarrow d_1 + d_2 \leq n_1. \quad \text{(D.11)}$$

B) $I_1 = d_3 - (m_3 - n_1)$, which occurs if $m_3 > n_1$ and $d_4 \leq m_4$, we have

$$\begin{aligned} I_1 + d_1 + d_2 \leq n_1 &\Leftrightarrow d_1 + d_2 + d_3 \leq m_3 \\ &\overset{m_3 \geq n_1}{\Leftrightarrow} d_1 + d_2 + d_3 \leq \max(m_3, n_1). \quad \text{(D.12)} \end{aligned}$$

TABLE VI
POSSIBLE VALUES FOR $\boldsymbol{I}_1$ IN THEOREM 10

Case	$I_1 =$
A	0
B	$d_3-(m_3-n_1)$
C	$d_3+d_4-(m_3+m_4-n_1)$
D	d_3
E	$d_3+d_4-(m_3+m_4-n_1)^+$

C) $I_1 = d_3 + d_4 - (m_3 + m_4 - n_1)$, which occurs if $m_3 > n_1$ and $d_4 > m_4$, we have

$$I_1 + d_1 + d_2 \leq n_1 \Leftrightarrow \sum d_i \leq m_3 + m_4 \overset{m_3 \geq n_1}{\Leftrightarrow} \sum d_i \leq \max(m_3 + m_4, n_1). \quad \text{(D.13)}$$

D) $I_1 = d_3$, which occurs if $m_3 \leq n_1$, we have

$$I_1 + d_1 + d_2 \leq n_1 \Leftrightarrow d_3 + d_1 + d_2 \leq n_1 \overset{m_3 \leq n_1}{\Leftrightarrow} d_1 + d_2 + d_3 \leq \max(m_3, n_1). \quad \text{(D.14)}$$

E) Finally, in case of $I_1 = d_3 + d_4 - (m_3 + m_4 - n_1)^+$, we have

$$I_1 + d_1 + d_2 \leq n_1 \Leftrightarrow \sum d_i \leq \max(m_3 + m_4, n_1). \quad \text{(D.15)}$$

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Authors' photographs and biographies not available at the time of publication.