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Comprehensive study on a 2 × 2 full-rate and linear decoding complexity space–time block code

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Abstract: This paper presents a comprehensive study on the Full-Rate and Linear-Receiver (FRLR) STBC proposed as a newly coding scheme with the low decoding complexity for a 2×2 MIMO system. It is shown that the FRLR code suffers from the lack of the non-vanishing determinant (NVD) property that is a key parameter in designing a full-rate STBC with a good performance in higher data rates, across QAM constellation. To overcome this drawback, we show that the existence of the NVD feature for the FRLR code depends on the type of the modulation. In particular, it is analytically proved that the FRLR code fulfills the NVD property across the PAM constellation but not for the QAM scheme. Simulation results show that, at a BER equal to 10^{-4}, utilising the PAM modulation for the FRLR-STBC, provides about 2 dB gain over a use of the QAM when the bandwidth efficiency is 6b/s/Hz. In addition, for the PAM constellation, the FRLR code significantly outperforms some existing full-rate STBCs. Finally, we utilise the moment generating function approach to derive an exact closed-form expression for the average error probability of the FRLR code with the BPSK modulation.

1 Introduction

Space–time coding (STC) has been recognised as an effective diversity method to combat the fading effect using multiple transmit/receive antennas in wireless multiple-input–multiple-output (MIMO) channels without bandwidth expansion [1]. Space–time block code (STBC) is among STC techniques which has a potential capability for achieving a full-diversity gain (reliable communication) with a lower complexity in the implementation [2–4]. To achieve a full-diversity gain, most STBC designs are based on the rank and the determinant criteria which are derived in [5]. The rank criterion implies that the difference matrix of any two distinct codewords of an STBC must be full rank. When the full rank property is fulfilled by an STBC, the minimum determinant among all possible determinants of difference matrices is defined as the coding gain or equivalently the determinant criterion. One challenge faced in some designed STBCs with the above criteria (e.g. [6]) is that the values of their coding gains vanish when the data transmission rate increases. Belfiore *et al.* [7] and Oggier *et al.* [8] define the non-vanishing determinant (NVD) property as a refinement criterion that determines whether the coding gain of an STBC remains constant when the constellation size increases.

STBCs with two transmit and two receive antennas are among different MIMO settings that have been widely utilised by wireless communications standards such as the IEEE 802.16e [9] and the IEEE 802.11n [10]. The first 2 × 2 STBC scheme is proposed by Alamouti [1] which allows the system to benefit from a linear processing when the maximum likelihood (ML) decoding algorithm is used at the receiver. Despite the simplicity of the Alamouti scheme, it cannot achieve the optimal diversity-multiplexing gain (DMG) tradeoff for more than one receive antenna [11]. The problem of designing a 2 × 2 STBC that achieves the optimal DMG tradeoff has been addressed in [7, 12–14]. However, for a constellation with size M, the schemes in [7, 12–14] suffer from a high decoding complexity that is of order $O(M^4)$ when the optimum receiver is utilised to estimate the transmitted symbols. Since the battery life in a typical wireless system is limited [15], most recent works have been focused on providing the NVD property in an optimum receiver with a low decoding complexity [16–22]. The proposed schemes in [16–18] that were further investigated and named as fast-decodable STBCs (FDSTBCs) in [23] use the well-known Alamouti scheme [1] in their structures to reduce the decoding complexity. Furthermore, it is shown in [23] that the computational complexity of FDSTBCs with the sphere decoder algorithm is at most of order $O(M^3)$ that is much less than $O(M^4)$ for the conventional STBCs with the NVD property in [7, 12–14] in particular, for high data transmission rates where M is large. In addition, the decoding complexity of the proposed scheme in [17] reduces to the order of $O(M^2)$ with utilising the conditional decoding method.

In our previous work [24], we proposed a 2 × 2 full-rate and linear-receiver (FRLR) STBC whose decoding complexity is of order $O(M)$ for the ML decoder. Numerical results in [24] show that the FRLR code has the best performance in terms of the bit error rate (BER) among full-rate STBCs with the binary phase shift keying (BPSK) constellation. However, the FRLR code suffers from the lack of the NVD property over high-order quadrature amplitude modulation (QAM) constellations because of the limited search domain in the parameters optimisation process. This fact motivates us to investigate whether or not the FRLR code can achieve the NVD property with other types of constellations.

In this paper, we aim to improve the BER performance of the FRLR-STBC in a 2×2 MIMO system for higher data rates or equivalently constellations with more number of points. Towards this goal, a comprehensive analysis, supported by some numerical results, is presented to study the NVD problem of the FRLR code for the pulse amplitude modulation (PAM) and QAM constellations. In addition, a systematic optimisation method, based on the 'coding gain' [5] and the 'union bound with the exact pairwise error probability (PEP)' [25] criteria, is utilised to find the optimum design parameters for the FRLR code. It is analytically proved that the FRLR code only achieves the NVD property across PAM constellations and provides a remarkable performance gain when compared with QAM constellations. Simulation results show that, at a BER equal to 10^{-4}, utilising the PAM modulation for the FRLR-STBC, provides about 0.5 dB gain over a use of the QAM modulation when the bandwidth efficiency is 4 b/s/Hz. Owing to the existence of the NVD property across the PAM constellation, this gain grows up to 2 dB when the bandwidth efficiency increases to 6 b/s/Hz. Besides, for the PAM modulation, the FRLR code significantly outperforms the Golden code [7] and the proposed codes in [16, 17] and provides the same performance with far lower complexity when compared with that of the code in [26]. To make a fair comparison, we also show that the FRLR code with PAM modulation scheme has a close BER performance in comparison with those of other full-rate STBCs with optimised constellation when bandwidth efficiencies are 4 and 6 b/s/Hz.

To complete our analysis, we utilise the moment generating function (MGF) approach to derive an exact closed-form expression for the average error probability of the FRLR code with the BPSK modulation. Since the derived error probability expression is a function of the signal-to-noise ratio (SNR) and the design parameters of the FRLR code, the exact average error probability formula of the FRLR code with the optimum design parameters is also obtained for all SNR ranges. It should be noted that the exact error probability problem of an STBC has been analysed from different perspectives (e.g. [27–30]) only for the orthogonal STBCs (OSTBCs) because there exists an equivalent single-input–single-output (SISO) model for OSTBCs. However, since the FRLR code has a more simplified structure among other 2×2 full-rate STBCs (e.g. [7, 12–14, 16, 17, 26]), an exact closed-form expression can be derived for the FRLR code with the BPSK modulation scheme.

The rest of this paper is organised as follows. Section 2 describes the system model including a 2×2 MIMO transmitter/receiver over a wireless channel. Section 3 is devoted to the structure of the FRLR code and its properties. In Section 4, we optimise the FRLR code for the PAM and QAM constellations based on the coding gain and the exact PEP criteria. Section 5 provides simulation evaluations with synthetic results and interpretation of the findings. In Section 6, we derive an exact closed-form formula for the average error probability of the FRLR code with the BPSK modulation. Finally in Section 7, an overview of the results and conclusions are presented.

Notations: Throughout this paper, we use normal letters for scalars. Matrices and column vectors are set in bold capital and lower-case letters, respectively. We denote $\boldsymbol{X} = [x_{i,j}]_{n \times m}$ as an $n \times m$ matrix with the entries $x_{i,j}$. We use $\boldsymbol{I}_n$ and $\boldsymbol{0}_n$ for an $n \times n$ identity and zeros matrix, respectively. $\|\boldsymbol{A}\|_F$ denotes the Frobenius norm of matrix $\boldsymbol{A}$. Operators $(.)^T$ and $(.)^*$ are the transposition and complex conjugation. Operator $\bar{(.)}$ stacks an $n \times m$ complex matrix $\boldsymbol{X}$ to a real column vector $\bar{\boldsymbol{x}} = [x_{1,1}^R\ x_{1,1}^I\ x_{2,1}^R\ x_{2,1}^I \ldots x_{n-1,m}^R\ x_{n-1,m}^I\ x_{n,m}^R\ x_{n,m}^I]^T$. The set of complex numbers, integers, Gaussian integers, odd and even Gaussian integers are denoted by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}[\boldsymbol{i}]$, $\mathbb{Z}^{\text{odd}}[\boldsymbol{i}]$ and $\mathbb{Z}^{\text{even}}[\boldsymbol{i}]$, respectively. A real and complex Gaussian random variables with mean μ and variance σ^2 are denoted by $\mathcal{N}(\mu, \sigma^2)$ and $\mathcal{CN}(\mu, \sigma^2)$, respectively. We occasionally use the notations $\Re\{.\}$ and $\Im\{.\}$ or the superscripts $(.)^R$ and $(.)^I$ to denote the real and the imaginary parts of a complex number.

2 System model

We consider a 2×2 MIMO system along with an STBC scheme that transmits four modulated symbols over two consecutive time slots in a quasi-static Rayleigh fading channel. A binary data stream of length 4 k is divided into four binary sub-streams $\mathcal{B}_i \triangleq (b_{i1}, \ldots, b_{ik})$, $i = 1, \ldots, 4$, where $b_{ij} \in \{0, 1\}$ and k is the size of $\mathcal{B}_i$. Our system utilises an M-ary modulation scheme including QAM and PAM, in which each sub-stream $\mathcal{B}_i$ is mapped into the symbol $s_i \in \mathbb{Z}^{\text{odd}}[\boldsymbol{i}]$. The STBC encoder takes four modulated signals $\{s_i\}_{i=1}^4$ as its inputs and generates the codeword matrix $\boldsymbol{X} = [x_{t,i}]_{2 \times 2}$ whose entries $\{x_{t,i}\}_{t,i=1}^2 \in \mathbb{C}$, under the average power constraint $\mathbb{E}[|x_{t,i}|^2] = E_s$, where E_s is the average energy of constellation symbols, are associated with the linear combination of symbols $\{s_i\}_{i=1}^4$ and their complex conjugates. Then, at the first and the second time slots, the first and the second rows of matrix $\boldsymbol{X}$ denoted by $\boldsymbol{x}_1 = [x_{t,1}]_{1 \times 2}$ and $\boldsymbol{x}_2 = [x_{t,2}]_{1 \times 2}$ are transmitted from two transmit antennas, respectively. Assuming $h_{i,j}$ is the channel coefficient between ith transmit and jth receive antennas, and $n_{t,j}$ is the white Gaussian noise sample of jth receive antenna at the tth time slot, the received signal at time tth over jth receive antenna is given by $y_{t,j} = \left(1/\sqrt{2E_s}\right)\sum_{i=1}^2 x_{t,i}h_{i,j} + n_{t,j}$. Gathering all the received signals $\{y_{t,j}\}_{t,j=1}^2$, $\{h_{i,j}\}_{i,j=1}^2$ and $\{n_{t,j}\}_{t,j=1}^2$ in 2×2 matrices $\boldsymbol{Y} = [y_{t,j}]_{2 \times 2}$, $\boldsymbol{H} = [h_{i,j}]_{2 \times 2}$ and $\boldsymbol{N} = [n_{t,j}]_{2 \times 2}$, respectively, results in

$$\boldsymbol{Y} = \frac{1}{\sqrt{2E_s}}\boldsymbol{XH} + \boldsymbol{N} \tag{1}$$

The entries of the channel matrix $\boldsymbol{H}$ are independent identically distributed (i.i.d.) complex Gaussian random variables with zero-mean and unit-variance symbolised as $\{h_{i,j}\}_{i,j=1}^2 \sim \mathcal{CN}(0, 1)$. It is assumed that the channel matrix $\boldsymbol{H}$ is constant over two consecutive time slots and is known at the receiver side but not at the transmitter side. This assumption is reasonable due to the fact that the channel variations are slow, and therefore the channel estimation can be easily performed by transmitting known pilots or training signals to the receiver [31, 32, p. 7]. Finally, the components of matrix $\boldsymbol{N}$, that is, $\{n_{t,j}\}_{t,j=1}^{2} \sim \mathcal{CN}(0, N_0)$, are i.i.d. complex Gaussian random variables with zero-mean and variance N_0.

In this paper, the coherent ML decoder, known as an optimum decoder, is utilised to recover the transmitted symbols. Using the matrix form in (1), the ML decoder minimises the following cost function

$$\boldsymbol{F}(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4; s_1, s_2, s_3, s_4) = ||\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{H}||_F^2 \quad (2)$$

over all possible quadruplets $(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4)$.

3 FRLR code construction and its properties

In this section, two equivalent matrices as the block structures of the FRLR code are introduced. We then investigate the conditions under which the FRLR code achieves the full-diversity and the linear decoding properties. In addition, we present a discussion on the existence of the cubic shaping and the NVD properties for the FRLR code. The structure of the FRLR code introduced in [24] is as follows

$$\boldsymbol{X} = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} = \begin{pmatrix} a_1 s_1 + a_2 s_2 & b_1 s_3 + b_2 s_4 \\ -a_1 s_3^* - a_2 s_4^* & b_1 s_1^* + b_2 s_2^* \end{pmatrix} \quad (3)$$

where $a_i \triangleq |a_i| \mathrm{e}^{\mathrm{j}\phi_i}$ and $b_i \triangleq |b_i| \mathrm{e}^{\mathrm{j}\psi_i}$, $i = 1, 2$. To investigate the cubic shaping property of the FRLR code, we first collect the coded symbols in the stacked complex vector $\boldsymbol{x} = [x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}]^{\mathrm{T}}$. Then, we decompose the elements of the vector $\boldsymbol{x}$ into their real and imaginary parts denoted by $x_{i,j}^{\mathrm{R}}$ and $x_{i,j}^{\mathrm{I}}$, and rewrite (3) to obtain the equivalent real-valued structure of the FRLR code as follows (see (4))

$$\bar{\boldsymbol{x}} \triangleq \begin{pmatrix} x_{1,1}^{\mathrm{R}} \\ x_{1,1}^{\mathrm{I}} \\ x_{2,1}^{\mathrm{R}} \\ x_{2,1}^{\mathrm{I}} \\ x_{1,2}^{\mathrm{R}} \\ x_{1,2}^{\mathrm{I}} \\ x_{2,2}^{\mathrm{R}} \\ x_{2,2}^{\mathrm{I}} \end{pmatrix} = \underbrace{\begin{pmatrix} a_1^{\mathrm{R}} & -a_1^{\mathrm{I}} & a_2^{\mathrm{R}} & -a_2^{\mathrm{I}} & 0 & 0 & 0 & 0 \\ a_1^{\mathrm{I}} & a_1^{\mathrm{R}} & a_2^{\mathrm{I}} & a_2^{\mathrm{R}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1^{\mathrm{R}} & -b_1^{\mathrm{I}} & b_2^{\mathrm{R}} & -b_2^{\mathrm{I}} \\ 0 & 0 & 0 & 0 & b_1^{\mathrm{I}} & b_1^{\mathrm{R}} & b_2^{\mathrm{I}} & b_2^{\mathrm{R}} \\ 0 & 0 & 0 & 0 & -a_1^{\mathrm{R}} & -a_1^{\mathrm{I}} & -a_2^{\mathrm{R}} & -a_2^{\mathrm{I}} \\ 0 & 0 & 0 & 0 & -a_1^{\mathrm{I}} & a_1^{\mathrm{R}} & -a_2^{\mathrm{I}} & a_2^{\mathrm{R}} \\ b_1^{\mathrm{R}} & b_1^{\mathrm{I}} & b_2^{\mathrm{R}} & b_2^{\mathrm{I}} & 0 & 0 & 0 & 0 \\ b_1^{\mathrm{I}} & -b_1^{\mathrm{R}} & b_2^{\mathrm{I}} & -b_2^{\mathrm{R}} & 0 & 0 & 0 & 0 \end{pmatrix}}_{\triangleq G} \underbrace{\begin{pmatrix} s_1^{\mathrm{R}} \\ s_1^{\mathrm{I}} \\ s_2^{\mathrm{R}} \\ s_2^{\mathrm{I}} \\ s_3^{\mathrm{R}} \\ s_3^{\mathrm{I}} \\ s_4^{\mathrm{R}} \\ s_4^{\mathrm{I}} \end{pmatrix}}_{\triangleq \bar{s}} = \boldsymbol{G}\bar{\boldsymbol{s}} \quad (4)$$

where $\boldsymbol{G}$ is called the real generator matrix of the FRLR code. Parameters $\{a_1, a_2, b_1, b_2 \in \mathbb{C}\}$ should be designed such that the low BER performance is achieved under the following constraints:

- *Constraint I*: Since symbols s_i's are drawn from the constellations with odd-integer coordinates, the parameters $\{a_1, a_2, b_1, b_2\}$ must satisfy $a_1 z_1 + a_2 z_2 \neq 0$ and $b_1 z_1 + b_2 z_2 \neq 0$, except for $z_1 = z_2 = 0$, where $z_1, z_2 \in \mathbb{Z}^{\mathrm{odd}}[\boldsymbol{i}]$. This constraint is a necessary condition for achieving the full-diversity gain by the FRLR code (see [24] for the proof).
- *Constraint II*: To guarantee that the equal average power is transmitted at each time slot, the coefficients $\{a_1, a_2, b_1, b_2\}$ must satisfy

$$|a_1|^2 + |a_2|^2 = |b_1|^2 + |b_2|^2 = 1 \quad (5)$$

- *Constraint III*:

$$a_1 b_2^* = a_2 b_1^* \quad \text{or} \quad a_1 a_2^* = b_1^* b_2 \quad (6)$$

This constraint reduces the complexity of the ML receiver to the linear complexity (see [24] and [33, Appendix I] for more details).

Obviously, the above constraints reduce the search region of the designed coefficients when performing the optimisation procedure. To see this fact, one can readily combine (5) and (6), and obtain to the following equation

$$|a_1|\sqrt{1 - |a_1|^2}\mathrm{e}^{\mathrm{j}\Delta\phi} = |b_1|\sqrt{1 - |b_1|^2}\mathrm{e}^{\mathrm{j}\Delta\psi} \quad (7)$$

where $\Delta\phi \triangleq \phi_1 - \phi_2$ and $\Delta\psi \triangleq \psi_2 - \psi_1$. By comparing the right- and the left-hand sides of (7), we conclude the following implicit constraints

$$|a_1| = |b_1| \quad \text{or} \quad |a_2| = |b_b| \quad (8)$$

$$\Delta\phi = \Delta\psi \quad (9)$$

Considering (5), (8) and (9) lead to the fact that there are only two degrees of freedom (DoFs), that is, $|a_1| = |b_1|$ (or $|a_2| = |b_2|$) and $\Delta\phi$ (or $\Delta\psi$), that can be found to optimise the BER performance of the FRLR code. The FRLR code also benefits from the full-rate property since the number of transmitted symbols is equal to the product of the number of time slots and the number of receive antennas [32, p. 12]. Moreover, the spectral efficiency of the FRLR code is two symbols per channel use or equivalently $2\log_2(M)$ bits per channel use (BPCU). Another examinable property of an STBC is the 'cubic shaping' that guarantees the average power of each coded symbol $x_{i,j}$ is independent from the correlation and the mean of input symbols [17]. In particular, when the input symbols s_i's have correlation and/

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or non-zero-mean, the average power of coded symbols $x_{i,j}$'s is more than that of the case where the correlation and the mean of input symbols are zero. Hence, for a non-cubic shaping scheme, the statistical properties of the input symbols impress the system performance. It is discussed in full detail in [32, p. 40] that if the generator matrix of an STBC, $\boldsymbol{G}_{m\times n}$, is an orthogonal matrix, that is, $\boldsymbol{G}^{\mathrm{T}}_{n\times m}\boldsymbol{G}_{m\times n} = \boldsymbol{I}_n$, the STBC satisfies the cubic shaping property. It is a straightforward computation to check the cubic shaping property of the FRLR code. In fact, from (4) we have

$$\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G} \neq \boldsymbol{I}_8 \tag{10}$$

It is seen that the FRLR code cannot satisfy the cubic shaping property. However, it is reasonable to assume that the input symbols s_i's are uncorrelated because of the enhanced interleaves in the overall system components [17]. Hence, the FRLR code preserves the same average power of the input symbols s_i's for the coded symbols $x_{i,j}$'s when it is utilised as a coding scheme in practical situations.

4 Parameters optimisation of FRLR code

In this section, we consider the BPSK constellation as the basic modulation scheme that provides the high coding gain 4 for the FRLR code as described in [24]. To use the FRLR code with a higher data rate, we want to extend the BPSK constellation through two following ways and investigate which one is more efficient in terms of the BER performance: (i) increase the number of points of the BPSK constellation in one-dimensional (1D) to obtain the PAM constellation and (ii) increase the number of points of the BPSK constellation to a 2D space to obtain the QAM constellation. Towards this goal, we first perform an in-depth analysis to optimise parameters of the FRLR code for the PAM constellation and show that it preserves the BPSK coding gain across the PAM constellation. Supported by numerical results, we then formally demonstrate that, unlike the PAM constellation, the FRLR's coding gain over the BPSK constellation vanishes when the constellation size increases to a 2D space.

4.1 Parameters optimisation for PAM constellation

By having the NVD property for the FRLR code across PAM constellations, in the next theorem, we find the optimum values of the implicit design parameters $|a_1|$ and $\Delta\phi$ in (8) and (9) such that the coding gain is independent from the constellation size.

Theorem 1: For a given PAM constellation, the maximum coding gain of the FRLR code is achieved when $|a_1|$ is equal to $1/\sqrt{2}$ and the parameter $\Delta\phi$ is within the range $(\pi/3) + k\pi \leqslant \Delta\phi \leqslant (2\pi/3) + k\pi$, $k \in \mathbb{Z}$. For this case, the value of coding gain is four and is independent from the constellation size.

Proof: Suppose that $\boldsymbol{X}$ and $\boldsymbol{X}'$ are two distinct codewords associated with the FRLR code. According to the coding gain criterion, we must maximise $|\det(\boldsymbol{X}-\boldsymbol{X}')|^2$ over all possible codewords $\boldsymbol{X}\neq\boldsymbol{X}'$, that is (see (11))

where $e_i = \left(s_i - s_i^{'}\right) \in \mathbb{Z}^{\mathrm{even}}$, $i = 1, \ldots, 4$. According to Constraint I in Section 3, the first (resp., the second) term in (11) is zero if and only if (iff) $e_1 = e_2 = 0$ (resp., $e_3 = e_4 = 0$). The third term of (11) is also zero iff $e_1 = e_2 = 0$ or $e_3 = e_4 = 0$. Applying Constraint III in Section 3 to (11) gives us (see (12))

Since all three terms in (12) are non-negative, the minimum determinant can be obtained when either $e_1 = e_2 = 0$ or $e_3 = e_4 = 0$. Assuming that $e_3 = e_4 = 0$, we have

$$\max_{a_1,a_2,b_1,b_2}\left\{\min_{e_1,e_2}\left\{\left|(a_1e_1 + a_2e_2)\left(b_1e_1^* + b_2e_2^*\right)\right|^2\right\}\right\} \tag{13}$$

Using Constraint III in Section 3 and (8) and (9), the expansion of (13) could be expressed as

$$\max_{\Delta\phi,|a_1|}\{\min_{|e_1|,|e_2|,\Delta\theta}\{[|a_1|^2|e_1|^2 + (1-|a_1|^2)|e_2|^2 + 2|a_1|\sqrt{1-|a_1|^2}|e_1||e_2|\cos(\Delta\phi+\Delta\theta)]^2\}\} \tag{14}$$

where $e_i = \left|e_i\right|\mathrm{e}^{\mathrm{j}\theta_i}$, $i = 1, 2$ and $\Delta\theta = \theta_1 - \theta_2$. Since $\Delta\theta$ only takes two values $\{0, \pi\}$ over PAM constellations, the minimum determinant of the FRLR code can be rewritten as follows

$$\max_{\Delta\phi,|a_1|}\{\min_{|e_1|,|e_2|}\{[|a_1|^2|e_1|^2 + (1-|a_1|^2)|e_2|^2 - 2|a_1|\sqrt{1-|a_1|^2}|e_1||e_2||\cos(\Delta\phi)|]^2\}\} \tag{15}$$

We note from (15) that the FRLR's coding gain is a periodic

$$\max_{a_1,a_2,b_1,b_2}\left\{\min_{e_1,e_2,e_3,e_4}\left|\det(\underbrace{\boldsymbol{X}-\boldsymbol{X}'}_{\triangleq\boldsymbol{E}})\right|^2\right\} = \max_{a_1,a_2,b_1,b_2}\left\{\min_{e_1,e_2,e_3,e_4}\left\{\left|(a_1e_1+a_2e_2)(b_1e_1^*+b_2e_2^*)\right|^2 + \left|(b_1e_3+b_2e_4)(a_1e_3^*+a_2e_4^*)\right|^2 + 2\Re\left\{(a_1e_1+a_2e_2)(b_1e_1^*+b_2e_2^*)(b_1^*e_3^*+b_2^*e_4^*)(a_1^*e_3+a_2^*e_4)\right\}\right\}\right\} \tag{11}$$

$$\max_{a_1,a_2,b_1,b_2}\left\{\min_{e_1,e_2,e_3,e_4}\left\{\left|(a_1e_1+a_2e_2)(b_1e_1^*+b_2e_2^*)\right|^2 + \left|(b_1e_3+b_2e_4)(a_1e_3^*+a_2e_4^*)\right|^2 + 2\Re\left\{\left[a_1b_1^*e_1e_3^* + a_1b_2^*(e_1e_4^*+e_2e_3^*) + a_2b_2^*e_2e_4^*\right]\left[a_1b_1^*e_1e_3^* + a_1b_2^*(e_1e_4^*+e_2e_3^*) + a_2b_2^*e_2e_4^*\right]^*\right\}\right\}\right\} \tag{12}$$

function of $\Delta\phi$ with the period π. Now, the following four cases could be considered for the simplified expression (15)

Case I: $|e_1| \neq 0,\ |e_2| = 0$: For this case, the minimum determinant of the FRLR code in (15) is simplified as

$$\min_{|e_1|}\left|\det(\boldsymbol{E})\right|^2 = \min_{|e_1|}\{|a_1|^4|e_1|^4\} = 16|a_1|^4 \quad (16)$$

Case II: ($|e_1| = 0,\ |e_2| \neq 0$): For this case, the minimum determinant of the FRLR code in (15) is simplified as

$$\begin{aligned}\min_{|e_2|}\left|\det(\boldsymbol{E})\right|^2 &= \min_{|e_2|}\{(1-|a_1|^2)^4|e_2|^4\}\\ &= 16(1-|a_1|^2)^4 \quad (17)\end{aligned}$$

Since (16) and (17) are independent of $\Delta\phi$, obviously, the maximum achievable coding gain over Cases I and II is four which is obtained when $|a_1| = 1/\sqrt{2}$. Hence, the optimum power allocation for providing the maximum coding gain of the FRLR code across PAM constellations is the uniform power scheme.

Case III: $|e_1| = |e_2| \neq 0$: In this case, we have

$$\begin{aligned}\min_{|e_1|}\left|\det(\boldsymbol{E})\right|^2 &= \min_{e_1}|e_1|^4(1-|\cos(\Delta\phi)|)^2\\ &= 16(1-|\cos(\Delta\phi)|)^2 \quad (18)\end{aligned}$$

where we use the optimum value of parameter $|a_1|$, that is, $1/\sqrt{2}$.

Case IV: $|e_1| = n|e_2|$(or $|e_2| = n|e_1|$), where n is a rational number greater than one:

For the case $|e_1| = n|e_2|$, we have

$$\min_{|e_1|,n}\left|\det(\boldsymbol{E})\right|^2 = \min_{n} 16\left(\frac{1+n^2}{2} - n|\cos(\Delta\phi)|\right)^2 \quad (19)$$

where we use the optimum value of $|a_1| = 1/\sqrt{2}$.

From (16) to (19), it turns out that the maximum achievable coding gain is 4. To obtain a minimum determinant equal or >4 by Case III, $\Delta\phi$ must lie within $(\pi/3) + k\pi \leqslant \Delta\phi \leqslant (2\pi/3) + k\pi,\ k \in \mathbb{Z}$. If this range could satisfy Case IV with a value equal or >4 then the proof will be completed.

Let us assume $(\pi/3) + k\pi \leqslant \Delta\phi \leqslant (2\pi/3) + k\pi, k \in \mathbb{Z}$, and consider Case IV. For this case

$$\left(\frac{1+n^2}{2} - n|\cos(\Delta\phi)|\right) \geq \frac{1}{2}(1 + n(n-1)) > \frac{1}{2} \quad (20)$$

thus

$$16\left(\frac{1+n^2}{2} - n|\cos(\Delta\phi)|\right)^2 > 4 \quad (21)$$

Therefore, selecting $\Delta\phi$ within the range $(\pi/3) + k\pi \leqslant \Delta\phi \leqslant (2\pi/3) + k\pi,\ k \in \mathbb{Z}$, maximises the coding gain. □

Remark 1: According to Theorem 1, there exist infinite number of $\Delta\phi$'s in the range of $(\pi/3) + k\pi \leqslant \Delta\phi \leqslant (2\pi/3) + k\pi,\ k \in \mathbb{Z}$, can maximise the FRLR's coding gain. However, as we will show in Section 5, the BER performance of the FRLR code is not the same for different values of $\Delta\phi$. We will use a more accurate performance metric to address this inconsistency in the next section.

Remark 2: It is worth mentioning that the values of the parameters obtained in Theorem 1 are also optimum for any rotated version of PAM constellations. Clearly, this result comes from the fact that $\Delta\theta$ and $|e_i|$'s take the same values $\{0, \pi\}$ and even integers, respectively, for a rotated PAM constellation with arbitrary angles. Therefore, following the proof steps in Theorem 1, one can simply prove that any rotation applied to the PAM constellation has no effect on the optimised values of design parameters in Theorem 1.

4.2 Parameters optimisation for QAM constellation

So far, we have obtained the optimum coding gain of the FRLR code for PAM constellations and verified its robustness against increasing the constellation size. In this section, we aim to demonstrate that this robustness is collapsed when the QAM constellation is utilised as the modulation scheme for the FRLR code. The next theorem will present our investigations for the FRLR's coding gain over the QAM constellations.

Theorem 2: The maximum achievable coding gain of the FRLR code over the QAM constellation is smaller than four.

Proof: For simplicity, we first prove the theorem for the 4-QAM constellation and then give a short discussion about the generalisation to other sizes of QAM constellation.

Recalling from (11) to (14) in the proof of Theorem 1, the coding gain relationship for the 4-QAM constellation can be expressed as

$$\begin{aligned}&\max_{a_1,a_2,b_1,b_2}\left\{\min_{e_1,e_2,e_3,e_4}|\det(\underbrace{\boldsymbol{X}-\boldsymbol{X}'}_{\triangleq\boldsymbol{E}})|^2\right\}\\ &= \max_{\Delta\phi,|a_1|}\{\min_{|e_1|,|e_2|,\Delta\theta}\{[|a_1|^2|e_1|^2 + (1-|a_1|^2)|e_2|^2\\ &\quad + 2|a_1|\sqrt{1-|a_1|^2}|e_1||e_2|\cos(\Delta\phi+\Delta\theta)]^2\}\} \quad (22)\end{aligned}$$

where $e_1,\ e_2 \in \mathbb{Z}^{\text{even}}[\boldsymbol{i}]$. Note that $|e_i|$ for $i = 1, 2$ and $\Delta\theta$ take values $\{0,\ 2,\ 2\sqrt{2}\}$ and $\{0, \pi,\ \pm(\pi/2),\ \pm(\pi/4)\}$, respectively, over the 4-QAM constellation. Similar to the arguments in Theorem 1, if $|a_1| \neq 1/\sqrt{2}$, the coding gain value is smaller than four and the theorem will be proved. Therefore, we choose $|a_1| = 1/\sqrt{2}$ and rewrite (22) as

$$\max_{\Delta\phi}\left\{\min_{|e_1|,|e_2|,\Delta\theta}\left\{\left[\frac{1}{2}(|e_1|^2+|e_2|^2) + |e_1||e_2|\cos(\Delta\phi+\Delta\theta)\right]^2\right\}\right\} \quad (23)$$

In the sequel, if we find a value smaller than four for (23) at least in a special case, the theorem will be proved. Now, let us assume $|e_1| = |e_2| \neq 0$ and rewrite (23) as

$$\max_{\Delta\phi}\{\min_{|e_1|,\Delta\theta}\{[|e_1|^2(1+\cos(\Delta\phi+\Delta\theta))]^2\}\} \quad (24)$$

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To show that (24) provides a value smaller than four, we consider four values of (24) corresponding to $\{|e_1| = 2, \Delta\theta = 0, \pi, \pi/2, -\pi/2\}$. Therefore our optimisation problem becomes as

$$\max_{\Delta\phi} \{\min\{16(1 - |\cos(\Delta\phi)|)^2, \ 16(1 - |\sin(\Delta\phi)|)^2\}\} \quad (25)$$

Obviously, in order to take a value ≥ 4 by the expression (25), $\Delta\phi$ has to be simultaneously within the ranges $((\pi/3) \leqslant |\Delta\phi| \leqslant (2\pi/3))$ and $(0 \leqslant |\Delta\phi| \leqslant (\pi/6)$ or $(5\pi/6) \leqslant |\Delta\phi| \leqslant \pi)$, which is not possible. Hence, the FRLR's coding gain is smaller than four over the 4-QAM constellation. Since 4-QAM is a basic structure of M-QAM constellation for $M = 2^q$, $q = 3, 4, \ldots$, therefore the above result can be readily generalised to higher sizes of QAM constellations. □

Remark 3: Unlike the PAM constellation, Theorem 2 states that increasing the data rate of the FRLR code through extending the BPSK constellation to a 2D space such as the QAM leads to decreasing its maximum achievable coding gain. It can be shown that there exists no lower bound on the FRLR's coding gain when the QAM constellation size increases. Experimental observations show that when $M^{\text{QAM}} \to \infty$, the number of values $\Delta\theta$ and α, where $\alpha \triangleq |e_2|/|e_1|$, increases such that $\Delta\theta$ and α take any value in ranges $[-\pi \ \pi]$ and [0 1], respectively. Therefore, for any choice of $\Delta\phi$, there exists a $\Delta\theta$ leads the argument of cos $(\Delta\theta + \Delta\phi)$ tends to $k\pi$, where $k \neq 0$ is an odd integer. Hence, we could write (22) when $M^{\text{QAM}} \to \infty$ as follows

$$\delta_{\min}^{\text{QAM}} \to \min_{|e_1|, |e_2|} \{[|a_1||e_1| - |a_2||e_2|]^4\} \quad (26)$$

Without loss of generality, we assume that $0 < |a_1| \leq (1/\sqrt{2})$ and $(1/\sqrt{2}) \leq |a_2| < 1$ and rewrite (26) as follows

$$\delta_{\min}^{\text{QAM}} \to \min_{|e_1|, \alpha} \left\{ |e_1|^4 |a_2|^4 \left[\frac{|a_1|}{|a_2|} - \alpha \right]^4 \right\} \quad (27)$$

Since $0 < |a_1|/|a_2| \leq 1$, there exists an α such that $\alpha \to |a_1|/|a_2|$. Therefore the coding gain of the FRLR code approaches to zero when $M^{\text{QAM}} \to \infty$.

Remark 4: It is good to give an intuitive interpretation of the fact that the FRLR code can reach the NVD property with the PAM constellation unlike the QAM constellation. The cosine argument $(\Delta\phi + \Delta\theta)$ in (14), which depends on the signal constellation through $\Delta\theta$ and the design parameter through $\Delta\phi$, plays a critical role whether the FRLR's coding gain is constant over a specific constellation or not. Intuitively, optimisation of $\delta_{\min}$ is equivalent to find an optimum $\Delta\phi$ which adjusts the cosine argument $(\Delta\phi + \Delta\theta)$ for different values of $\Delta\theta$ such that the coding gain is maximised. Obviously, when $\Delta\theta$ takes a few values on a specific constellation, the adjustment of $\Delta\phi$ along with a high coding gain is possible. For example, the high coding gain 4 is achieved for the PAM constellation with any arbitrary size by choosing the design parameter $\Delta\phi$ within the range $(\pi/3) + k\pi \leqslant \Delta\phi \leqslant (2\pi/3) + k\pi, \ k \in \mathbb{Z}$. On the other hand, for the case where $\Delta\theta$ takes more values, achieving a high coding gain by adjusting the cosine argument $(\Delta\phi + \Delta\theta)$ is an impossible task. For example, consider 4-QAM constellation where $\Delta\theta$ takes values $\{0, \pi, \pm(\pi/2), \pm(\pi/4)\}$.

Table 1 Optimum parameters of the FRLR code for the QAM constellation

Constellation	Average power distribution	$\lvert a_1\rvert = \lvert b_1\rvert$	$\lvert a_2\rvert = \lvert b_2\rvert$	$\Delta\phi$	$\delta_{\min}$
4-QAM	**non-uniform**	**0.46**	**0.8897**	**2.88**	**0.7117**
4-QAM	uniform	0.7071	0.7071	5.76	0.2863
8-QAM	**non-uniform**	**0.7500**	**0.6614**	**5.17**	**0.1904**
8-QAM	uniform	0.7071	0.7071	4.27	0.1483

To achieve the high coding gain 4, $\Delta\phi$ must simultaneously satisfy the intervals $((\pi/3) + k\pi \leqslant \Delta\phi \leqslant (2\pi/3) + k\pi)$ and $(0 \leqslant |\Delta\phi| \leqslant (\pi/6)$ or $(5\pi/6) \leqslant |\Delta\phi| \leqslant \pi)$ associated with $\Delta\theta = 0$, π and $\Delta\theta = \pm(\pi/2)$, respectively, which is not possible. Therefore the maximum achievable coding gain becomes a value smaller than four. This scenario is still repeated when the size of constellation increases and the value of the coding gain dramatically decreases.

Remark 5: Owing to the increase in the values of the parameter $\Delta\theta$ for higher sizes of the QAM constellation, the optimisation procedure of the FRLR's parameters should be separately performed over different sizes of the QAM constellation. To avoid the complication of the analytical method, we use the numerical search to find the optimum values for $|a_1|$ (or $|b_1|$) and $\Delta\phi$ (or $\Delta\psi$) on ranges (0, 1) and (0, 2π), respectively. It is worth mentioning that in contrast to the PAM constellation, the uniform distribution of the average power over different symbols s_i's is not an optimum solution for the QAM constellation. In fact, the determinant of the optimised FRLR code for a given QAM is not minimum when ($|e_1| \neq 0$, $|e_2| = 0$) or ($|e_2| \neq 0$, $|e_1| = 0$) which cannot necessitate the uniform average power distribution over symbols s_i's as the optimum solution. For instance, for the 4-QAM constellation, the minimum determinant of the optimised FRLR code is obtained when $|e_1| = |e_2| = 2$ and $\Delta\theta = 0$. Table 1 gives the optimum parameters and the coding gain of the FRLR code for the 4-QAM and the 8-QAM constellations with the non-uniform and the uniform average power distributions over symbols s_i's. It is seen from Table 1 that for a given QAM constellation, the FRLR code with the non-uniform average power distribution has the best solution in terms of the coding gain when compared with the uniform power case.

5 Simulation results and discussion

In this section, we present some simulation results to validate the theoretical results obtained in Section 4. In addition, we compare the BER performance of the FRLR code for the PAM constellation with that of the best full-rate STBCs in the literature. The simulation setup consists of a MIMO system with two transmit and two receive antennas. We assume that the average power of each symbol s_i, $i = 1, \ldots, 4$, the variance of the channel coefficients $h_{i,j}$'s are normalised to unit, and the variance of the noise samples $n_{i,j}$'s is set to N_0. The channel matrix is assumed to be constant over two consecutive time slots and changes independently from one coherence interval to the next. We deploy the practical modulations PAM and QAM along with the Gray mapping in our simulations. In each simulation run and for a fixed SNR, a frame of random bits of length $L_d = 10\,000$ is generated and then modulated to $L_d/\log 2(M)L$ symbol frames. Each symbol frame is coded by the STBC encoder and then transmitted over the random channel generated by the MATLAB program. After transmitting and decoding all symbol frames, a condition is checked whether the number of errors in a frame of decoded bits is reached to $0.1L_d$ or not. If it is reached, the simulation running will be stopped and the average BER will be obtained by taking an average over all channel realisations. Otherwise, the simulation running continues till the condition is reached. In addition, the BER performance of our proposed FRLR code is compared with those of the following STBCs schemes: (i) the Golden code [7], (ii) the Paredes–Gershman–Alkhansari (PGA) code [16], (iii) the Sezginer–Sari–Biglieri (SSB) code [17] and (iv) the Ghaderipoor–Hajiaghayi–Tellambura (GHT) code [26].

5.1 PAM constellation results

Fig. 1 illustrates the BER performance of the FRLR code against SNR with the BPSK and PAM constellations for some $\Delta\phi$'s in the optimum range $(\pi/3) + k\pi \leqslant \Delta\phi \leqslant (2\pi/3) + k\pi$, $k \in \mathbb{Z}$, obtained in Theorem 1. It is seen that although there are an infinite number of $\Delta\phi$'s that maximise the FRLR's coding gain for PAM constellations, the FRLR code does not exhibit the same BER performance for different values of $\Delta\phi$'s. In particular, the worst case for the BER performance is for the boundaries points $\Delta\phi = (\pi/3) + k\pi$ or $(2\pi/3) + k\pi$, whereas the best one is in the middle range, that is, $\Delta\phi = (\pi/2) + k\pi$. It is worth mentioning that although the effect of the design parameter $\Delta\phi$ seems negligible on the BER performance of the FRLR code, our aim in this paper is to find the best solution regarding this performance metric for the FRLR code. Generally, the error probability of the FRLR code decreases when $\Delta\phi$ increases from $(\pi/3) + k\pi$ to $(\pi/2) + k\pi$ and the behaviour is reversed when $\Delta\phi$ grows from $(\pi/2) + k\pi$ to $(2\pi/3) + k\pi$.

In an attempt to verify the above inconsistency, we use the union bound with the exact PEP metric given in [25], denoted by P_{ue}, that is more accurate but complicated than the coding gain and is utilised as the design criterion of the GHT code [26] (see [25] for more details and [26] for a thorough review of the method). Table 2 displays the values of P_{ue} at some sample SNRs, denoted by P_{ue}^{SNR}, for different values of $\Delta\phi$. As we observe from Table 2, for a specific SNR, the value of the metric P_{ue}^{SNR} is in a complete agreement with the extracted results from Fig. 1. In particular, when $\Delta\phi$ grows from $(\pi/3) + k\pi$ to $(\pi/2) + k\pi$, the values of P_{ue}^{SNR}'s decrease and the behaviour is reversed from $(\pi/2) + k\pi$ to $(2\pi/3) + k\pi$. In addition, the best (i.e. the minimum) and the worst (i.e. the maximum) values of P_{ue}^{SNR} correspond to $\Delta\phi = (\pi/2) + k\pi$ and $\Delta\phi = (\pi/3) + k\pi$, respectively.

Remark 6: According to the above results for the PAM constellation, our optimisation method for the FRLR code can be systematically framed as a pseudo global–local search in the following two steps:

- *Step I*: Using the coding gain criterion, a primary range is analytically found for the parameter $\Delta\phi$ (global search). In this step, the search space of $\Delta\phi$ is reduced using a simple analysis based on the coding gain criterion.
- *Step II*: For the obtained range in Step I, the metric P_{ue} is numerically utilised to find the optimum solution of $\Delta\phi$ (local search). In fact, the optimum solution is found by applying the more accurate but complicated criterion P_{ue} over a smaller range of $\Delta\phi$.

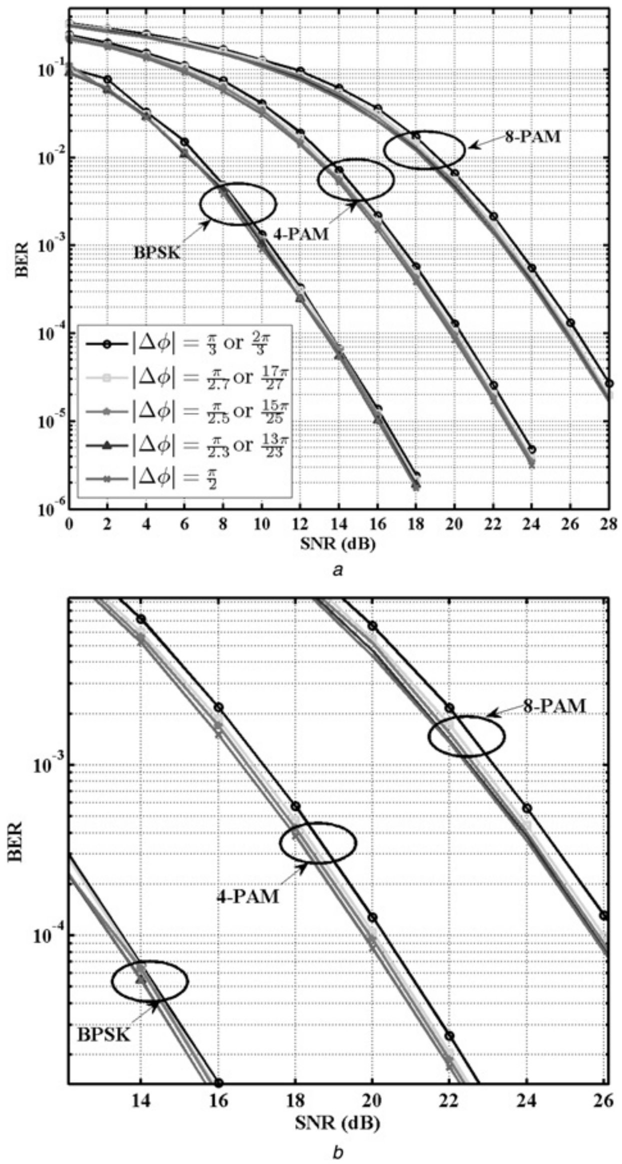

Fig. 1 *BER against SNR of the FRLR code for different values of $\Delta\phi$, and for the BPSK and PAM constellations*

a Full view
b Zoomed-in view of a part of Fig. 1*a*

5.2 Performance comparison between PAM and QAM constellations

In this section, we compare the BER performance of the FRLR code for the PAM and QAM constellations. In all simulations, we use $\{|a_1| = 1/\sqrt{2},\ \Delta\phi = (\pi/2) + k\pi\}$ and the optimised parameters in Table 1 for the PAM and QAM constellations, respectively. For simplicity in our simulation illustrations, we call a QAM constellation with the uniform (resp., non-uniform) average power distribution as the uniform (resp., non-uniform) QAM constellation.

Fig. 2 compares the BER performance of the FRLR code for the PAM and QAM constellations. It is seen from Fig. 2 that the FRLR code has the best BER performance in all

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Table 2 Metric P_{ue} for different values of $\Delta\phi$'s for the BPSK and PAM constellations

Constellation	$\Delta\phi + k\pi$	$\frac{\pi}{3}$	$\frac{\pi}{2.7}$	$\frac{\pi}{2.5}$	$\frac{\pi}{2.3}$	$\frac{\pi}{2}$	$\frac{13\pi}{23}$	$\frac{15\pi}{25}$	$\frac{17\pi}{27}$	$\frac{2\pi}{3}$
BPSK	$P_{ue}^{14} \times 10^3$	0.9861	0.8431	0.7804	0.7390	0.7139	0.7390	0.7804	0.8431	0.9861
	$P_{ue}^{16} \times 10^3$	0.2124	0.1797	0.1660	0.1573	0.1521	0.1573	0.1660	0.1797	0.2124
	$P_{ue}^{18} \times 10^4$	0.4155	0.3490	0.3221	0.3052	0.2954	0.3052	0.3221	0.3490	0.4155
4-PAM	$P_{ue}^{16} \times 10^2$	0.87	0.72	0.65	0.60	0.57	0.60	0.65	0.72	0.87
	$P_{ue}^{20} \times 10^3$	0.430	0.344	0.308	0.284	0.270	0.284	0.308	0.344	0.430
	$P_{ue}^{24} \times 10^4$	0.149	0.118	0.105	0.097	0.092	0.097	0.105	0.118	0.149
8-PAM	P_{ue}^{20}	0.056	0.046	0.041	0.038	0.035	0.038	0.041	0.046	0.056
	$P_{ue}^{24} \times 10^4$	34	27	24	22	20	22	24	27	34
	$P_{ue}^{28} \times 10^7$	1367	1061	933	851	802	851	933	1061	1367

ranges of SNR when the PAM modulation is utilised. More precisely, at a BER equal to 10^{-4}, the FRLR code with the PAM constellation outperforms the non-uniform QAM constellation by about 0.5 and 2 dB when the BPCU is 4 and 6 b/s/Hz, respectively. As a result, the gain achieved by the PAM constellation with respect to the QAM constellation grows when the BPCU or equivalently the constellation size increases. This result is expected because the NVD property achieves when the PAM constellation is used as the modulation scheme for the FRLR code but this is not true for the QAM constellation (see Table 1). In addition, although the BER performance of the FRLR code with the non-uniform 4-QAM constellation is superior to that with the uniform 4-QAM constellation, it gives the same performance for the uniform and non-uniform 8-QAM constellations. This result reflects the coding gains differences between the non-uniform QAM and the uniform QAM that are big, that is, 0.4254 and small, that is, 0.0421 for 4- and 8-QAM constellations, respectively (see Table 1).

5.3 Performance comparison with other full-rate STBCs

For completeness, Fig. 3 shows the BER performance against SNR of the optimised FRLR code and that of the Golden [7], GHT [26], PGA [16] and SSB [17] codes for the BPSK and PAM constellations. As can be seen, both the FRLR and the GHT codes display exactly the same performances and significantly outperform the Golden, PGA and SSB codes in all SNR ranges. To verify this result, we also compare the coding gain values of the codes under simulation in Table 3. We can see that the coding gain of the FRLR and the GHT codes is higher than that of other STBC schemes. However, the FRLR code has an advantage over the GHT code in terms of the decoding complexity. As shown in [24], the decoding complexity of the FRLR code is linear with respect to the constellation size, that is, $O(M)$ but that grows as the fourth-power of the constellation size, that is, $O(M^4)$ for the GHT code. Another point of interest in

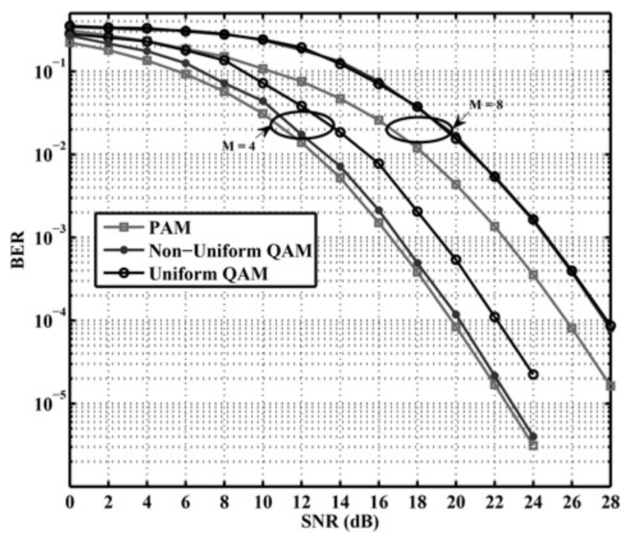

Fig. 2 *BER performance comparison between the PAM and QAM constellations for the FRLR code*

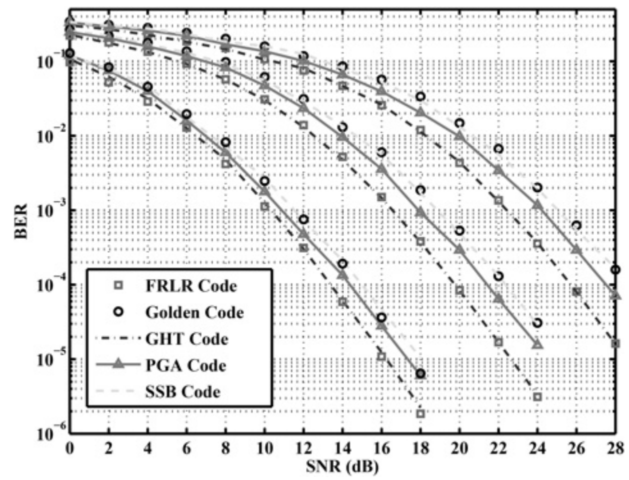

Fig. 3 *BER performance comparison between different STBCs for the PAM constellation*

Table 3 Coding gain of different STBCs for the BPSK and PAM constellations

Code	δ_{min}
FRLR	**4**
Golden	3.2
GHT	**4**
PGA	2.2857
SSB	2

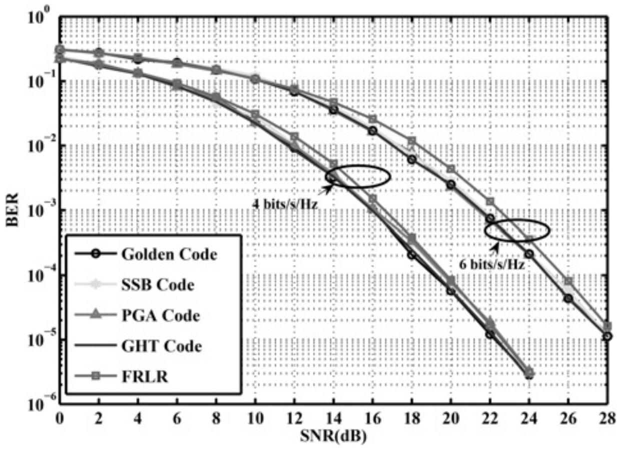

Fig. 4 *BER performance comparison between different STBCs with different modulation schemes*

Table 3 is the BER of the PGA code which is better than that of the Golden code, whereas the coding gain of the PGA code is lower than that of the Golden code. This inconsistency comes from the fact that $\delta_{\min}$ is a loose upper bound on the error probability of an STBC. To address this issue, one can compute the metric P_{ue} which is tighter than $\delta_{\min}$, for the Golden and PGA codes.

To make a fair comparison, we compare the BER performance of the Golden, the GHT, the PGA and the SSB codes when they utilise the QAM constellation as the modulation scheme with that of the FRLR code when it uses the PAM constellation in Fig. 4. Note that since the BER performances of the Golden, the GHT, the PGA and the SSB are same for the QAM modulation, the BER curves of these codes are coincided in Fig. 4. It is seen that the FRLR code with the PAM constellation has a close BER performance to those of other STBCs with the QAM constellation when the BPCU is 4 and 6 bits/s/Hz. For example, the difference value of the BER for the FRLR code and other STBCs is 0.5×10^{-5} in the SNR = 28 dB with BPCU = 6 bits/s/Hz. This little difference could be considered as a penalty for the complexity reduction since the optimised parameters of the FRLR code are selected from the complex region with extra conditions used for having a linear complexity.

6 Error probability of the FRLR code with BPSK modulation

In this section, we use the MGF approach (see [34, 35]) to derive an exact closed-form formula for the error probability of the FRLR code with the BPSK constellation. There exist several works devoted to find an exact closed-form expression for the error probability of OSTBCs in various fashions (e.g. [27–29, 36]). However unlike OSTBCs, since the MIMO model of other STBCs cannot be transformed to an equivalent SISO model, deriving an exact closed-form formula for other STBCs is not an easy task. In particular, for a general 2×2 full-rate STBC matrix, the number of conditional error probability expressions is the fourth-power of constellation size, that is, M^4. Fortunately, since the ML metric of the FRLR code is decoupled, similar to that of quasi-OSTBCs [37], to two independent expressions (as explained in [33, Appendix I]), this number reduces to $2M^2$ for the FRLR code. This fact makes possible an exact derivation for the error probability formula of the FRLR code at least over low-order modulation schemes such as BPSK. To the best of our knowledge, this is the first time that a closed-form expression is being developed for a full-rate STBC with a special modulation scheme. It should be noted that although the number of conditional error probabilities is small for the BPSK constellation, the analysis of each conditional error probabilities is still a tedious and cumbersome task (see [33, Appendix II]). Hence, the error probability analysis for high-order modulation schemes is a hard task and beyond the scope of the current work and is left for the future work.

Similar to the arguments in [27, 29, 35], our derivation approach for the error probability formula comprises the following two steps: (i) the conditional error probability of a message symbol given that the channel matrix and two received symbols is found as some non-proper integrals, (ii) the non-proper integrals are transformed into proper integrals, where the proper integral defines as $\int_{c_1}^{c_2} \exp\left\{-c_3 \frac{1}{\sin^2 \phi}\right\} \mathrm{d}\phi$ with finite constants c_i's, $i = 1, 2, 3$, using some change of variables. Then, we remove the condition by taking an average with respect to the channel matrix distribution. Since the average is taken with respect to the proper integrals, the result can be interpreted as finite integrals of MGF [34, 35]. Fortunately, the solutions of these integrals exist in terms of exact finite summations in the literature.

Owing to the limited space, the results are contained in the following theorem and the proof details could be found in [33, Appendix II].

Theorem 3: The average error probability of the FRLR code with the BPSK modulation scheme over i.i.d Rayleigh fading channel is given by (see (28) at the bottom of the next page)

where $\rho = 2\Re\{a_1 a_2^*\} = 2|a_1||a_2|\cos(\Delta\phi)$ is the correlation parameter (see [33, Appendix I]), and for $-\pi \le \psi \le \pi$ [38, Appendix] (see (29) at the bottom of the next page)

where n is positive integer, $\mu = \sqrt{c/(1+c)}\mathrm{sgn}(\psi)$, $\omega = -\mu \cot \psi$ and

$$T_{ik} = \frac{\binom{2k}{k}}{\binom{2k-2i}{k-i} 4^i (2k - 2i + 1)}$$

.

Proof: See [33, Appendix II]. □

Remark 7: It is worth mentioning that the average error probability of the FRLR code with the BPSK modulation, in addition to SNR, depends on the design parameters of the FRLR code, that is, $|a_1|$, $|a_2|$ and $\Delta\phi$ through the correlation parameter ρ. According to Theorem 1, the correlation coefficient range for achieving maximum coding gain of the FRLR code is $|\rho| \le 1/2$. Since the parameter ρ is zero for the optimum values of the design parameters, that is, $|a_1| = |a_2| = 1/\sqrt{2}$ and $\Delta\phi = \pi/2$, the average error

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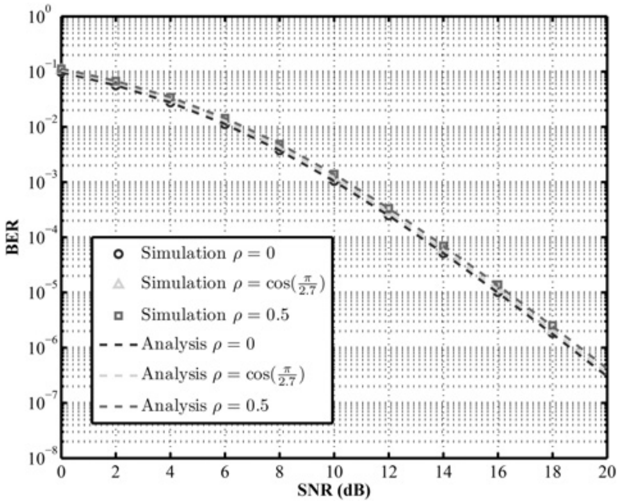

Fig. 5 $\mathbb{P}_e$ *against average SNR per receive antenna for the FRLR code with the BPSK modulation*

probability for this case simplifies as

$$\mathbb{P}_{\rm e}^{\rm opt} = I_4\left(\frac{1}{4N_0},\ \frac{\pi}{2}\right) \tag{30}$$

where $\mathbb{P}_{\rm e}^{\rm opt}$ denotes the average error probability of the FRLR code with the BPSK modulation when the optimum parameters are chosen. Fig. 5 compares the simulation result with the analysis method of the FRLR's BER over the BPSK modulation when the parameter ρ is 0, $\cos(\pi/2.7)$ and 0.5. It is seen that the derived closed-form error probability formula from Theorem 1 is exactly in agreement with the simulation results and again reveals that the optimum error probability for the FRLR code with the BPSK modulation achieves when the parameter $\rho = 0$ or equivalently when the design parameter $\Delta\phi = \pi/2$. In addition, the average BER performance of the FRLR code changes insignificantly with the change of correlation coefficient ρ. This result is expected since $\Delta\phi$ has a negligible impact on the average BER of the FRLR code (see Fig. 1) and it has a close relationship with the correlation coefficient, that is, $\rho^{\rm PAM} = \cos(\Delta\phi)$.

7 Conclusions

In this paper, we have presented a comprehensive study on the performance analysis of the FRLR-STBC. In the first part of this paper, it was analytically, supported by some simulation results, shown that the FRLR code only satisfies the NVD property across the PAM constellation but not for the QAM constellation. Simulation results revealed that the gain achieved by the FRLR code with the PAM modulation over the QAM case expectantly increases because of the existence of the NVD property across the PAM constellation. In the second part of this paper, using the MGF approach, an exact closed-form formula was derived for the error probability of the FRLR code with the BPSK modulation in terms of the SNR and the design parameters. To the best of our knowledge, this is the first time that an exact closed-form relationship has been formulated for a full-rate STBC with a special modulation scheme.

We believe that since a low decoding complexity is provided by the FRLR code, these results may be used for designing other such codes. The pseudo global–local search, as a new technique, could be utilised for parameters optimisation of other STBCs. In addition, this is the first time that a closed-form expression has been formulated for the average error probability of a full-rate STBC. This may lead to further research in deriving formula for average error probability of other full-rate STBCs.

$$\begin{aligned}
\mathbb{P}_{\rm e} = \frac{1}{4}\Bigg[& I_4\left(\frac{(1+2|\rho|)^2}{4N_0},\ \tan^{-1}\left\{\sqrt{\frac{1+|\rho|}{1-|\rho|}}\right\}\right) - I_4\left(\frac{1-|\rho|}{2N_0},\ \tan^{-1}\left\{\sqrt{\frac{1}{\rho^2}-1}\right\}\right) + I_4\left(\frac{(1+2|\rho|)^2}{4N_0},\ \frac{\pi}{2}\right) + I_4\left(\frac{1}{4N0},\ \frac{\pi}{2}\right)\Bigg] \\
& + \frac{1}{4}\left[I_4\left(\frac{(1-2|\rho|)^2}{4N_0},\ \tan^{-1}\left\{\frac{1-2|\rho|}{1+2|\rho|}\sqrt{\frac{1+|\rho|}{1-|\rho|}}\right\}\right) + I_4\left(\frac{1-|\rho|}{2N_0},\ \tan^{-1}\left\{-\sqrt{\frac{1}{\rho^2}-1}\right\}+\pi\right)\right] \\
& - \frac{1}{8}\left[I_4\left(\frac{(1+2|\rho|)^2}{4N_0},\ \frac{\pi}{2}+\tan^{-1}\left\{\frac{|\rho|}{\sqrt{1-\rho^2}}\right\}\right) + I_4\left(\frac{(1+2|\rho|)^2}{4N_0},\ \frac{\pi}{2}-\tan^{-1}\left\{\frac{|\rho|}{\sqrt{1-\rho^2}}\right\}\right)\right] \\
& + \frac{3}{8}\left[I_4\left(\frac{1}{4N_0},\ \frac{\pi}{2}-\tan^{-1}\left\{\frac{|\rho|}{\sqrt{1-\rho^2}}\right\}\right) + I_4\left(\frac{1}{4N_0},\ \frac{\pi}{2}+\tan^{-1}\left\{\frac{|\rho|}{\sqrt{1-\rho^2}}\right\}\right)\right] \\
& - \frac{1}{2}\left[I_4\left(\frac{1}{4N_0},\ \tan^{-1}\left\{\sqrt{\frac{1+|\rho|}{1-|\rho|}}\right\}\right)\right]
\end{aligned} \tag{28}$$

$$\begin{aligned}
I_n(c,\ \psi) &\triangleq \frac{1}{\pi}\int_0^{\psi}\left(\frac{\sin^2\phi}{\sin^2\phi + c}\right)^n {\rm d}\phi \\
&= \frac{\psi}{\pi} - \frac{\mu}{\pi}\left[\left(\frac{\pi}{2}+\tan^{-1}\omega\right)\sum_{k=0}^{n-1}\binom{2k}{k}\frac{1}{[4(1+c)]^k} + \sin(\tan^{-1}\omega)\sum_{k=1}^{n-1}\sum_{i=1}^{k}T_{ik}\frac{[\cos(\tan^{-1}\omega)]^{2(k-i)+1}}{(1+c)^k}\right]
\end{aligned} \tag{29}$$

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