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Belief propagation-based multiuser receivers in optical code-division multiple access systems

Mohammad Ali Sedaghat[1], *Alireza Nezamalhosseini*[1], *Hamid Saeedi*[2], *Farokh Marvasti*[1]

[1]*Department of Electrical Engineering Department, Sharif University of Technology, Tehran, Iran*
[2]*Department of Electrical and Computer Engineering Department, Tarbiat Modares University, Tehran, Iran*
E-mail: mohammad.sedaghat@iet.ntnu.no

Abstract: In this study, the authors investigate the performance of optical code-division multiple access (OCDMA) systems with belief propagation (BP)-based receivers. They propose three receivers for the optical fibre channel that provide a trade-off between detecting complexity and system performance. The first proposed receiver achieves a performance very close to the so-called known interference lower bound. The second receiver exhibits a considerably less complexity at the expense of a slight degradation in performance. They show that the third BP-based receiver, which is a simplified version of the second receiver, is surprisingly the same as the so-called multistage detector in OCDMA systems. They then study the problem of finding proper spreading codes for the proposed receivers. BP-based receivers perform well if the graph corresponding to the spreading matrix has no short cycles. The probability of existence of short cycles directly depends on the sparsity of the spreading matrix. Therefore they look for sparse spreading matrices that are also uniquely detectable, that is, the corresponding input data vectors and the output spread vectors are in one-to-one correspondence. The existence of random uniquely detectable matrices (for which the elements are binary with equal probability) has already been proved by Edrös and Rényi when the dimensions of matrix tend to infinity. In this study, they prove the existence of sparse uniquely detectable spreading matrices in the large system limit, when the number of users and the number of chips approach infinity and their ratio is kept constant. For finite length systems, they propose to use optical codes with one chip interference between codes and show that they exhibit a better performance than random sparse codes.

1 Introduction

Code-division multiple access (CDMA) is a well-known multiple access technique which has extensive application in wireless and optical communication systems. In a CDMA system, all resources such as time and frequency are available to all users simultaneously. Each user is assigned a unique spreading code (signature) to be distinguished from other users. Consequently, any CDMA system is represented by a matrix called the spreading matrix. Each column of the spreading matrix is a spreading code. [Throughout the paper, these two terms, spreading matrix and spreading codes, are exchangeably.] The transmitted CDMA signal can be represented as the multiplication of the data of different users by the spreading matrix. In fibre-optic networks, optical CDMA (OCDMA) can be considered as a high data rate multiple access scheme because of its ability to establish a robust multiple-access environment for a number of users [1].

OCDMA can be used in both asynchronous and synchronous scenarios. Asynchronous OCDMA does not require any time synchronisation among users. In this regard, OCDMA systems with optical orthogonal codes (OOCs) were introduced by Salehi *et al.* [1–6]. Another set of codes for asynchronous OCDMA are known as prime codes (PCs) [7]. For both schemes, simplicity at the receiver is achieved at the expense of degrading the spectral efficiency compared with other multiple access techniques such as time-division multiple access. In other words, the maximum number of users in asynchronous OCDMA is considerably smaller than the number of chips. In synchronous OCDMA systems, the time synchronisation between users is required. For synchronous OCDMA, modified PC (MPCs) and transpose MPC (T-MPC) are proposed in the literature, which are developed based on PC [8, 9].

The optimal detector in a CDMA system is computationally complex to implement and therefore many suboptimal receivers have been proposed for CDMA systems [10]. For OCDMA systems in particular, they include the correlator receiver, correlator with hard limiter, chip level detector, multistage detector, successive and parallel interference cancellers, and expectation maximisation-based multiuser detection [11–17].

Recently, a new method for multiuser detection, called sparsely spread CDMA, has emerged in the literature [18–24]. This technique utilises belief propagation (BP) algorithm with near-optimal performance, which is a lot more computationally efficient compared with the optimal detection in [10]. The main idea is to use the sparse spreading code that is inspired by the successful deployment of the BP decoding in low-density parity-check (LDPC) codes.

The methods in [18–24] are developed based on the assumption of a channel with additive white Gaussian noise. Moreover, the receiver is assumed to know channel state information. Therefore in optical CDMA systems where there are other sources of uncertainty such as Poisson channel and also some non-additive noise sources such as dark current noise, the receivers proposed in [18–24] cannot be directly applied.

In this paper, we propose a BP-based scheme that is devised particularly for OCDMA systems. The proposed BP-based scheme includes three receivers with different complexity/performance specifications. We show that the first proposed receiver can achieve a performance very close to the known interference lower bound. In this lower bound, it is assumed that all users know the information bits of other users [11]. In the second algorithm, we propose to use an approximated version of the BP algorithm to considerably reduce the complexity at the expense of slight degradation in the performance compared with the first receiver. In the third proposed receiver, we use some more approximations to the algorithm in the second receiver to further reduce the complexity. This causes more performance degradation. Surprisingly, we observe that the third receiver, which is still a simplified version of the BP algorithm, is in fact identical to the multistage detector introduced in [11].

The main contribution of this paper is to provide algorithms with practical complexity that perform reasonably well in the overloaded region. For example, in a system with 64 chips over ideal fibre, the first proposed algorithm can support as high as 86 users with a bit error rate (BER) as low as 10^{-3}. The performance of other implementation-efficient algorithms in the literature is far from that of our scheme in both over- and underloaded regions.

We also study the choice of codes for the proposed receivers to have the best performance. In general, there are two sources of errors in the BP-based receivers in addition to the channel noise. The first source of error is the cycles in the factor graph representation of the system [25] and the second one is the existence of multiple user data vectors for a given spread vector. The first source of error can be removed by choosing random sparse matrices based on asymptotic cycle free (ACF) property [26] to avoid cycles with length shorter than a desired value. The second one can be removed if the spreading matrix is uniquely detectable [By uniquely detectable matrix, we mean an injective matrix, that is, the inputs and outputs are in one-to-one correspondence.] [27–29]. It is notable that the second source of error is more dominant in the overloaded systems, where the loading factor, that is, the ratio of the number of users to the number of chips, is >1. To remove both sources of error, we have to look for sparse spreading matrices that are also uniquely detectable. The existence of random uniquely detectable matrices (for which the elements are binary with equal probability) is proven by Edrös and Rényi in [30] when the dimension of matrix tends to infinity. Moreover, based on the results of [31], under certain conditions, the so-called ‘optimum asymptotic multiuser efficiency’ for random i.i.d. codes is 1 in the large system limit [By the large system limit, we mean the case where the number of users and the number of chips approach infinity while their ratio is kept constant.], which also implies the existence of uniquely detectable spreading matrices. However, the conditions in [31] are not satisfied in our case where the random sparse unipolar matrices are used. In this paper, we rigorously prove that a random sparse spreading matrix chosen based on the ACF property can be uniquely detectable almost surely in the large system limit.

Our results on the existence of uniquely detectable sparse spreading matrices are valid in the large system limit. In the finite length systems with random sparse codes, short cycles in the factor graph are inevitable. Thus, we propose to use a different class of codes based on OOC [1, 2] for which we can avoid cycles of length four. We show that such codes outperform finite length random codes.

The rest of this paper is organised as follows. Section 2 gives a description of the OCDMA system model. In Section 3, maximum a posteriori probability (MAP) detector in an OCDMA system and its factor graph representation are described. In Section 4, the BP-based receivers for multiuser detection in optical fibre channels are presented. Section 5 is dedicated to the choice of proper codes for the proposed receivers. Section 6 presents simulation results in the optical fibre channel and Section 7 concludes the paper.

2 System model

We consider a bit synchronous OCDMA system in which users transmit signals simultaneously with on–off keying (OOK) modulation. It is important to note that this assumption is only for simplicity and the proposed scheme can be generalised to asynchronous OCDMA systems. In this system, each user has a unique spreading code of length L and the signal of the ith user in 1 bit duration time can be written as

$$x_i(t) = X_i \sum_{l=1}^{L} c_l^i p_{T_c}(t - lT_c), \quad 0 \le t \le T_b = LT_c \tag{1}$$

where X_i is the ith user's binary data, c_l^i is the lth component of the spreading code, T_c is the chip time interval, T_b is the bit period and $p_{T_c}(t)$ is a rectangular pulse defined as

$$p_{T_c}(t) = \begin{cases} 1, & 0 \le t \le T_c \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

In the noiseless channel with K users, the received signal at nth time interval ($nT_b \le t \le (n+1)T_b$) can be expressed as

$$y(t) = \sum_{i=1}^{K} \sum_{l=1}^{L} c_l^i X_i p_{T_c}(t - lT_c - nT_b) \tag{3}$$

This model can be considered in the ideal optical fibre channel in which the performance of the system is limited only by the multiple access interference. In such a channel, the effects of optical sources and the photodetector and the effects of all sources of noise are not considered. Therefore the sampled received signal can be written in discrete matrix form as

$$\boldsymbol{Y} = \boldsymbol{C}\boldsymbol{X} \tag{4}$$

where $\boldsymbol{C}$ is a binary matrix whose columns are the spreading code and $\boldsymbol{X}$ is a binary column vector, where $\boldsymbol{X} = [X_1, \ldots, X_K]^T$ and $\boldsymbol{Y} = [Y_1, \ldots, Y_L]^T$ is the received vector.

In a non-ideal optical fibre channel, we must consider the effect of noise and the channel model. We consider a

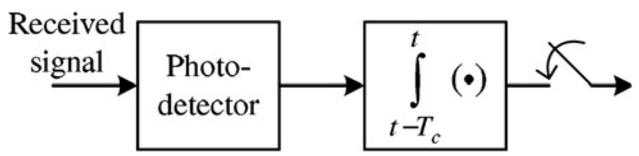

Fig. 1 *Block diagram of receiver in each chip*

receiver as shown in Fig. 1 that includes a photodetector and an integrator for each chip. We sample the output at the end of each chip. The sampled output for the ith user at the lth chip can be modelled by a Poisson distribution with mean $X_i\lambda_x T_c$, where

$$\lambda_x = \frac{2}{w} P_{av} \frac{h}{hv} \tag{5}$$

In (5), P_{av}, λ_x, η, v and h indicate, respectively, the average received signal power on the photodetector area, the photo-electron count rate of the received signal, quantum efficiency, optical frequency and the Planck constant. Parameter w is the weight of spreading codes which will be defined more precisely later. We consider the effect of dark current noise and neglect the effects of other sources of noise such as thermal noise and beat noise. Dark current of photodetector can be modelled as a Poisson process with mean $\lambda_n T_c$, where

$$\lambda_n = \frac{i_{dc}}{q} \tag{6}$$

In (6), λ_n, i_{dc} and q are the photoelectron count rate of dark current noise, the dark current of photodetector and electron charge, respectively. Hence, the received signal with K users in the lth chip can be modelled by a Poisson random variable with mean $(m_l(\boldsymbol{X})\lambda_x + \lambda_n)T_c$, where $m_l(\boldsymbol{X})$ is the number of users which send '1' in the lth chip duration. In this case, the lth element of vector $\boldsymbol{Y}$ defined in (4) has a Poisson distribution with mean $(m_l(\boldsymbol{X})\lambda_x + \lambda_n)T_c$.

3 MAP detection and its factor graph representation

Let vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ be defined as in (4) and assume that we have received vector $\boldsymbol{Y}$ and want to estimate X_i, the ith element of $\boldsymbol{X}$, given $\boldsymbol{Y}$. Individually optimum MAP detector estimates X_i by maximising its a posteriori probability as

$$X_i = \arg \max_{X_i \in \{0,1\}} P\left(X_i | \boldsymbol{Y}\right) \tag{7}$$

Equation (7) can be written based on the marginal functions of the joint a posteriori probability as

$$\begin{aligned} X_i &= \arg \max_{\alpha \in \{0,1\}} \sum_{\boldsymbol{X} \in \{0,1\}^K, X_i = \alpha} P(\boldsymbol{X}|\boldsymbol{Y}) \\ &= \arg \max_{\alpha \in \{0,1\}} \sum_{\boldsymbol{X} \in \{0,1\}^K, X_i = \alpha} \frac{P(\boldsymbol{Y}|\boldsymbol{X})P(\boldsymbol{X})}{P(\boldsymbol{Y})} \end{aligned} \tag{8}$$

By eliminating the constant term $P(\boldsymbol{Y})$ from the denominator, the MAP rule can be simplified as

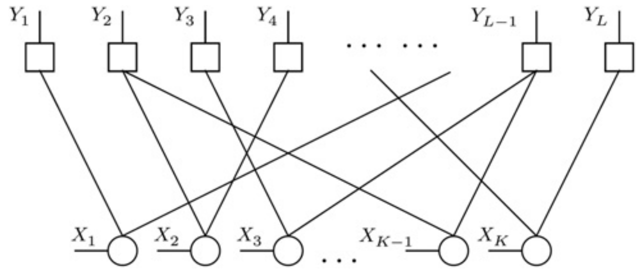

Fig. 2 *Factor graph model of the bit synchronous OCDMA system*

$$\begin{aligned} X_i &= \arg \max_{\alpha \in \{0,1\}} \sum_{\boldsymbol{X} \in \{0,1\}^K, X_i = \alpha} P(\boldsymbol{Y}|\boldsymbol{X})P(\boldsymbol{X}) \\ &= \arg \max_{\alpha \in \{0,1\}} \sum_{\boldsymbol{X} \in \{0,1\}^K, X_i = \alpha} P\left(\left\{Y_1, \ \ldots, \ Y_L\right\} | \boldsymbol{X}\right) P(\boldsymbol{X}) \end{aligned} \tag{9}$$

It can be seen that given $\boldsymbol{X}$, Y_l's are independent of each other and therefore $P(\boldsymbol{Y}|\boldsymbol{X}) = \prod_l P\left(Y_l|\boldsymbol{X}\right)$. Moreover, X_i's are independent of each other and hence we have

$$X_i = \arg \max_{\alpha \in \{0,1\}} \sum_{\boldsymbol{X} \in \{0,1\}^K, X_i = \alpha} \prod_l P\left(Y_l|\boldsymbol{X}\right) \prod_k P\left(X_k\right) \tag{10}$$

which shows the MAP detection can be converted into finding marginal functions of the following function [25]

$$f(\boldsymbol{X}, \boldsymbol{Y}) = \prod_l P(Y_l|\boldsymbol{X}) \prod_i P(X_i) \tag{11}$$

Marginal functions of $f(\boldsymbol{X}, \boldsymbol{Y})$ can be found by the BP algorithm on the factor graph model [25]. A factor graph is a bipartite graphical representation of a mathematical relation between some variable nodes and some factor nodes. We can realise the factor graph representation of an OCDMA scheme as depicted in Fig. 2. Each user bit X_k is represented by a circle, called the variable node or the symbol node. Each received signal entry Y_l represented by a square, called the factor node or the chip node. For any k and l, symbol node k and chip node l are connected by an edge if $C_{l,k}$, the element in lth row and the kth column of matrix $\boldsymbol{C}$, is equal to 1. Consequently, the MAP multiuser detection of OCDMA can be efficiently implemented using the BP algorithm.

4 Proposed BP-based multiuser receivers for OCDMA systems

In this section, three BP-based multiuser receivers for OCMDA systems over an optical fibre channel are proposed. The BP algorithm can be performed on the factor graph representation as follows. Each iteration of the BP algorithm has two halves. In the first half iteration, messages that are in the form of probabilities are sent from symbol nodes to chip nodes. Each chip node then computes the output message based on the received messages and sends it back to the symbol nodes to be used in the second half iteration. Therefore the BP algorithm at tth iteration can

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be written as [20]

$$P^t\left(X_k|\left\{Y_{j\neq l}\right\}\right) = \beta^t_{l,k}\prod_{j\neq l}P^t\left(Y_j|X_k,\left\{Y_{\mu\neq j}\right\}\right) \tag{12}$$

$$P^{t+1}\left(Y_l|X_k,\left\{Y_{j\neq l}\right\}\right) = \alpha^t_{l,k}\sum_{\{X_{m\neq k}\}}P\left(Y_l|\boldsymbol{X}\right) \times \prod_{m\neq k}P^t\left(X_m|\left\{Y_{j\neq l}\right\}\right) \tag{13}$$

where $\alpha^t_{l,k}$ and $\beta^t_{l,k}$ are constants for normalisation and $\{\boldsymbol{Y}_{j\neq l}\}$ is a vector of all of Y_j's excluding Y_l. $P^t(X_k/\cdot)$ [or $P^t(Y_k/\cdot)$] indicates the conditional probability that X_k (or Y_k) has a certain value in tth iteration. A posteriori probability, $P^t(X_k/\boldsymbol{Y})$, can be obtained as

$$P^t(X_k|Y) = \alpha_k\prod_j P^t\left(Y_l|X_k,\left\{Y_{j\neq l}\right\}\right) \tag{14}$$

Since we consider OOK signalling, the probabilities in (12) and (13) have to be computed twice for $X_k = 0$ and $X_k = 1$. Instead of doing so, we can use log-likelihood ratio value of the messages as

$$V^t_{k\to l} = \ln\left(\frac{P^t\left(X_k = 1|\left\{Y_{j\neq l}\right\}\right)}{P^t\left(X_k = 0|\left\{Y_{j\neq l}\right\}\right)}\right) \tag{15}$$

$$V^t_{l\to k} = \ln\left(\frac{P^t\left(Y_l|X_k = 1,\ \left\{Y_{j\neq l}\right\}\right)}{P^t\left(Y_l|X_k = 0,\ \left\{Y_{j\neq l}\right\}\right)}\right) \tag{16}$$

where $V^t_{k\to l}$ represents the message from symbol node k to chip node l and $V^t_{l\to k}$ represents the message from chip node l to symbol node k at the tth iteration. From (15) and (16), the first half iteration of the BP algorithm, (12), can be written as

$$V^t_{k\to l} = \sum_{j\in\partial k\backslash l}V^t_{j\to k} \tag{17}$$

where ∂k denotes the subset of chips connected directly to user k which is referred to as neighbourhood of the symbol node k. Also, let $\partial k\backslash l$ denote the neighbourhood of symbol node k excluding chip node l. Then, the second step of the algorithm in the lth chip can be written as (see (18))

The conditional probability distribution of Y_l, $P(Y_l|X_k, \{X_t\}_{t\in\partial l/k})$, is Poisson with mean $\lambda_l(\boldsymbol{X})T_c$ and is given by

$$P\left(Y_l|X_k,\ \{X_t\}_{t\in\partial l/k}\right) = \frac{\exp\left(-\lambda_l(\boldsymbol{X})T_c\right)\left(\lambda_l(\boldsymbol{X})T_c\right)^{Y_l}}{(Y_l)!} \tag{19}$$

where $\lambda_l(\boldsymbol{X}) = m_l(\boldsymbol{X})\lambda_x + \lambda_n$ in which $m_l(\boldsymbol{X})$ is the number of users which send 1 in the lth chip. In the rest of the paper for simplicity, we show $\lambda_l(\boldsymbol{X})$ and $m_l(\boldsymbol{X})$ by λ_l and m_l, respectively, and without loss of generality set $T_c = 1$. By substituting (19) into (18), we have

$$V^{t+1}_{l\to k} = \ln\left(\frac{A}{B}\right) \tag{20}$$

where

$$A = \sum_{\left\{X_{t\in\partial l/k}\right\},X_k=1}\left[\frac{\exp(-\lambda_l)(\lambda_l)^{Y_l}}{(Y_l)!}\right]\left[\prod_{j\in\partial l/k,X_j=1}\exp\left(V^t_{j\to l}\right)\right] \tag{21}$$

and

$$B = \sum_{\left\{X_{t\in\partial l/k}\right\},X_k=0}\left[\frac{\exp(-\lambda_l)(\lambda_l)^{Y_l}}{(Y_l)!}\right] \times\left[\prod_{j\in\partial l/k,X_j=1}\exp\left(V^t_{j\to l}\right)\right] \tag{22}$$

By eliminating $(Y_l)!$ from numerator and denominator and using the approximation

$$\ln\left(\sum_i\exp(r_i)\right) \simeq \max_i(r_i) \tag{23}$$

(20) can be written as

$$V^{t+1}_{l\to k} = \max_{\substack{\left\{X_{t\in\partial l/k}\right\},\\X_k=1}}\left(Y_l\ln(\lambda_l) - \lambda_l + \sum_{j\in\partial l/k,X_j=1}V^t_{j\to l}\right) - \max_{\substack{\left\{X_{t\in\partial l/k}\right\},\\X_k=0}}\left(Y_l\ln(\lambda_l) - \lambda_l + \sum_{j\in\partial l/k,X_j=1}V^t_{j\to l}\right) \tag{24}$$

We can summarise the operations in Algorithm 1 (see Fig. 3):

Two maximisations in the fourth step of Algorithm 1 (Fig. 3) in each iteration are done between 2^{ζ_l-1} terms in the lth chip, where ζ_l is the number of variable nodes connected to the lth chip node. If the values of ζ_l/K are small for all $l\in\{1, \ldots, L\}$ and the algorithm converges with finite number of iterations, the algorithm uses a number of operations that is linear in the number of chips. However, Algorithm 1 is more complicated than many conventional algorithms in OCDMA systems. In fact, the second stage of the BP algorithm [(18)] makes it too complicated. Now, we try to simplify (18) to obtain a simpler algorithm. It can be seen that (18) can be converted to

$$V^{t+1}_{l\to k} = \ln\left(\frac{\sum_{I^t_{lk}\in\{0,\ldots,\zeta_l-1\}}P\left(Y_l|X_k = 1,\ I^t_l\right)P\left(I^t_{lk}\right)}{\sum_{I^t_{lk}\in\{0,\ldots,\zeta_l-1\}}P\left(Y_l|X_k = 0,\ I^t_{lk}\right)P\left(I^t_{lk}\right)}\right) \tag{25}$$

$$V^{t+1}_{l\to k} = \frac{\sum\left\{X_{i\in\partial l/k}\right\}P\left(Y_l|X_k = 1,\quad\left\{X_{t\in\partial l/k}\right\}\right)\left[\Pi_{i\in\partial l/k}P^t\left(X_i|\left\{Y_{j\neq l}\right\}\right)\right]}{\sum\left\{X_{i\in\partial l/k}\right\}P\left(Y_l|X_k = 0,\quad\left\{X_{t\in\partial l/k}\right\}\right)\left[\Pi_{i\in\partial l/k}P^t\left(X_i|\left\{Y_{j\neq l}\right\}\right)\right]} \tag{18}$$

Initialisation:

1. **for all** l, k **do** $V_{l\to k} \leftarrow 0$.

Main iterations:

2. **for** $t = 1$ to T (the number of iteration) **do** 3,4.
3. **for all** k , $l \in \partial k$ **do**

$$V^t_{k\to l} = \sum_{j\in\partial k\backslash l} V^t_{j\to k}. \tag{25}$$

4. **for all** l , $k \in \partial l$ **do**

$$V^{t+1}_{l\to k} = \max_{\substack{\{X_{t\in\partial l\backslash k}\},\\ X_k=1}} \left(Y_l \ln(\lambda_l) - \lambda_l + \sum_{j\in\partial l\backslash k, X_j=1} V^t_{j\to l}\right) - \max_{\substack{\{X_{t\in\partial l\backslash k}\},\\ X_k=0}} \left(Y_l \ln(\lambda_l) - \lambda_l + \sum_{j\in\partial l\backslash k, X_j=1} V^t_{j\to l}\right) \tag{26}$$

5. **Return** $V_k = \sum_{l\in\partial k} V^T_{l\to k}$.
6. **If** $V_k \geq 0$ **then** $X_k = 1$, **else** $X_k = 0$.

Fig. 3 *Algorithm 1*

where

$$P\left(I^t_{lk}\right) = \prod_{\sum_{u\in\partial l\backslash k} X_u = I^t_{lk}} P^t\left(X_u | \left\{Y_{j\neq l}\right\}\right) \tag{26}$$

One can interpret I^t_{lk} as the total interference caused by all users excluding the kth user. From (26), it can be seen that I^t_{lk} is a Poisson Binomial random variable [The Poisson Binomial distribution is a discrete probability distribution of a sum of independent Bernoulli trials with unequal probabilities.] which can be approximated by a Gaussian distribution $N\left(\mu^t_{lk}, \sigma^2_{lkt}\right)$ in the large system limit, where

$$\begin{aligned}\mu^t_{lk} &= \sum_{u\in\partial l/k} P^t\left(X_u = 1 | \left\{Y_{j\neq l}\right\}\right)\\ &= \sum_{u\in\partial l/k} \frac{\exp\left(V^t_{u\to l}\right)}{\exp\left(V^t_{u\to l}\right) + 1}\end{aligned} \tag{27}$$

and

$$\sigma^2_{lkt} = \sum_{u\in\partial l/k} P^t\left(X_u = 1 | \left\{Y_{j\neq l}\right\}\right) \left(1 - P^t\left(X_u = 1 | \left\{Y_{j\neq l}\right\}\right)\right) = \sum_{u\in\partial l/k} \frac{\exp\left(V^t_{u\to l}\right)}{\left(\exp\left(V^t_{u\to l}\right) + 1\right)^2} \tag{28}$$

Equations (27) and (28) can also be approximated by two simple functions as

$$\mu^t_{lk} = \sum_{u\in\partial l\backslash k} U\left(V^t_{u\to l}\right) \tag{29}$$

and

$$\sigma^2_{lkt} = \sum_{u\in\partial l\backslash k} F\left(V^t_{u\to l}\right) \tag{30}$$

respectively, where

$$U(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases} \tag{31}$$

and

$$F(x) = \begin{cases} -0.09|x| + 0.29, & -3.22 \leq x \leq 3.22 \\ 0, & \text{otherwise} \end{cases} \tag{32}$$

By substituting Gaussian approximation of $P\left(I^t_{lk}\right)$ and (19) into (25) and using approximation (23), we have (see (33))

$$V^{t+1}_{l\to k} = \left\{-\left(\lambda_n + I^t_{lk}\lambda_x + \lambda_x\right) + Y_l \ln\left(\lambda_n + I^t_{lk}\lambda_x + \lambda_x\right) - \frac{\left(I^t_{lk} - \mu^t_{lk}\right)^2}{2\sigma^2_{lkt}}\right\}_{I^t_{lk}=I^{t1}_{l}} - \left\{-\left(\lambda_n + I^t_{lk}\lambda_x\right) + Y_l \ln\left(\lambda_n + I^t_{lk}\lambda_x\right) - \frac{\left(I^t_{lk} - \mu^t_{lk}\right)^2}{2\sigma^2_{lkt}}\right\}_{I^t_{lk}=I^{t0}_{lk}} \tag{33}$$

where I^{t1}_{lk} and I^{t0}_{lk} maximise the first and the second terms in the right-hand side of (33), respectively. Therefore I^{t1}_{lk} and I^{t0}_{lk} can be obtained by setting derivatives of the first and the second terms in the right-hand side of (33) equal to zero, respectively. Then, by some simplifications we have

$$\begin{aligned}&-\lambda_x\sigma^2_{lkt}\left(\lambda_n + I^{t1}_{lk}\lambda_x + \lambda_x\right) + Y_l\lambda\sigma^2_{lkt}\\ &\quad - \left(\lambda_n + I^{t1}_{lk}\lambda_x + \lambda_x\right)\left(I^{t1}_{lk} - \mu^t_{lk}\right) = 0\end{aligned} \tag{34}$$

and

$$-\lambda_x\sigma_{lkt}^2\left(\lambda_n + I_{lk}^{t0}\lambda_x\right) + Y_l\lambda\sigma_{lkt}^2 - \left(\lambda_n + I_{lk}^{t0}\lambda_x\right)\left(I_{lk}^{t0} - \mu_{lk}^t\right) = 0 \tag{35}$$

Equations (34) and (35) are two quadratic equations that can be solved easily. One can simplify (33) more by assuming large average rate of photoelectron (λ_x) as

$$V_{l\to k}^{t+1} = \left(I_{lk}^{t0} - I_{lk}^{t1}\right)\lambda_x + Y_l \ln\frac{\left(\lambda_n + I_{lk}^{t1}\lambda_x + \lambda_x\right)}{\left(\lambda_n + I_{lk}^{t0}\lambda_x\right)} \tag{36}$$

To simplify more, we approximate $V_{k\to l}^t$ by $V_k^t = \sum_{l\in\partial k} V_{l\to k}^t$. This assumption is valid in the large system limit. Therefore, in each iteration of the algorithm, only K messages $\left(V_k^t\right)$ have to be updated. Moreover, we should update μ_{lk}^t and σ_{lkt}^2 in each iteration with much less complexity than the update process in step 4 of Algorithm 1 (see Fig. 3). The proposed simplified algorithm can be summarised in Algorithm 2 (see Fig. 4):
we will show in simulation results that Algorithm 2 (Fig. 4) performs pretty closely to Algorithm 1 despite the considerable reduction in its complexity.

We now make some more simplification to Algorithm 2 which results in a new algorithm called Algorithm 3. To do so, we assume that $\sigma_{lkt}^2 \ll 1$, therefore the solutions of (41) and (42) (see Fig. 4) can be easily obtained as

$$I_{lk}^{t1} = \mu_{lk}^t \tag{37}$$

and

$$I_{lk}^{t0} = \mu_{lk}^t \tag{38}$$

Hence, step 4 in Algorithm 2 can be simplified by substituting (37) and (38) into it as

$$V_k^{t+1} = \sum_{l\in\partial k} -\lambda_x + Y_l \ln\left(1 + \frac{1}{\left(\lambda_n/\lambda_x\right) + \mu_{lk}^t}\right) \tag{39}$$

As a result, step 3.2 and the variance update in step 3.1 are removed from Algorithm 2. In this new algorithm, two variables V_k^t and μ_{lk}^t have to be updated. It can be observed that Algorithm 3 is the same as the well-known multistage detector introduced in [11]. In other words, the multistage detector can be considered as a simplified version of the BP algorithm. It is important to remind that this result is based on the assumption that $\sigma_{lkt}^2 \ll 1$ and it can be easily seen that in the beginning of the algorithm, this assumption is not valid. For example, in the first iteration, it can be seen from (28) that

$$\sigma_{lk1}^2 = \frac{(\zeta_l - 1)}{4} \tag{40}$$

Initialisation:

1- **for all** k **do** $V_k^0 \leftarrow 0$.

Main iterations:

2- **for** $t = 0$ to $T - 1$ (T is the number of iteration.) **do** 3, 4

3- **for** $l = 1$ to L and $k \in \partial l$ **do** 3.1, 3.2

Mean and Variance calculation:

3.1-

$$\mu_{lk}^t = \sum_{u\in\partial l\backslash k} U(V_u^t), \tag{39}$$

$$\sigma_{lkt}^2 = \sum_{u\in\partial l\backslash k} F(V_u^t), \tag{40}$$

Interference approximation:

3.2- **Solve**

$$-\lambda_x\sigma_{lkt}^2(\lambda_n + I_{lk}^{t1}\lambda_x + \lambda_x) + Y_l\lambda\sigma_{lkt}^2 - (\lambda_n + I_{lk}^{t1}\lambda_x + \lambda_x)(I_{lk}^{t1} - \mu_{lk}^t) = 0, \tag{41}$$

and

$$-\lambda_x\sigma_{lkt}^2(\lambda_n + I_{lk}^{t0}\lambda_x) + Y_l\lambda\sigma_{lkt}^2 - (\lambda_n + I_{lk}^{t1}\lambda_x)(I_{lk}^{t0} - \mu_{lk}^t) = 0. \tag{42}$$

4- **for all** k , **do**

$$V_k^{t+1} = \textstyle\sum_{l\in\partial k} (I_{lk}^{t0} - I_{lk}^{t1})\lambda_x + Y_l \ln\frac{(\lambda_n + I_{lk}^{t1}\lambda_x + \lambda_x)}{(\lambda_n + I_{lk}^{t0}\lambda_x)}. \tag{43}$$

5- **If** $V_k^T \geq 0$ **then** $X_k = 1$, **else** $X_k = 0$.

Fig. 4 *Algorithm 2*

where ζ_l is the degree of the lth chip. Equation (40) clearly shows that the assumption $\sigma^2_{lkt} \ll 1$ is not valid. As iterations continue, the value of σ^2_{lkt} starts to decrease making this assumption valid for larger number of iterations. This inconsistency causes the performance of Algorithm 3 to be worst than the two former algorithms.

To summarise, Algorithms 1, 2 and 3 (see Figs. 3 and 4) can provide a trade-off between complexity and performance of OCDMA detection.

5 Spreading code design

It is known that the BP algorithm performs optimally on graphs that are cycle free. However, for some of the most important applications of the BP algorithm such as decoding of LDPC codes, the underlying factor graph has cycles. Extensive simulation results show that for graphs with no short cycle, the BP algorithm can achieve astonishing performance [25]. Consequently, in graphs corresponding to OCDMA systems, we must choose the ones with no short cycles. In order to satisfy this condition, the spreading matrix should be sparse. In sparse matrices, if the average number of non-zero elements in each column of spreading matrix grows as slow as $o(L^{1/4t})$ with L (the number of chips), the probability that a variable node is involved in a cycle of length shorter than t approaches zero in the large system limit [26]. This is often called the ACF property. From ACF property, it can be observed that the number of short cycles decreases if the weight of spreading codes decreases. However, when the weight of codes decreases, the transformation between the user information bits, $\boldsymbol{X}$, and the channel output (in the noiseless channel), $\boldsymbol{CX}$, may not be one-to-one anymore. This is particularly more probable in the overloaded case, where the number of users is greater than the number of chips.

In general, the spreading matrix $\boldsymbol{C}$ is called uniquely detectable if there is a unique solution for $\boldsymbol{Y} = \boldsymbol{CX}$, that is

$$\boldsymbol{CX}_1 = \boldsymbol{CX}_2 \Rightarrow \boldsymbol{X}_1 = \boldsymbol{X}_2 \tag{41}$$

For OCMDA systems with OOK modulation, we have X_1, $X_2 \in \{0, 1\}^K$. Moreover, we can write (41) as follows

$$\boldsymbol{C}(\boldsymbol{X}_1 - \boldsymbol{X}_2) = 0 \Rightarrow \boldsymbol{X}_1 - \boldsymbol{X}_2 = 0 \tag{42}$$

By defining $\boldsymbol{Z} = \boldsymbol{X_1} - \boldsymbol{X_2} \in \{\pm 1, 0\}^K$, one can represent (42) as

$$\boldsymbol{CZ} = 0 \Rightarrow \boldsymbol{Z} = 0 \tag{43}$$

Another way to show (43) is [27–29]

$$\text{null}(\boldsymbol{C}) \bigcap \{\pm 1, 0\}^K = \phi \tag{44}$$

where null($\boldsymbol{C}$) is the null space of $\boldsymbol{C}$ and ϕ indicates the empty set. In [30], authors prove that random uniquely detectable matrices (when the entries of matrix are chosen randomly from {0, 1} with equal probability) exist even for large loading factor when the dimensions of matrix tend to infinity. Moreover, in [31], it is proved that under certain conditions, the so-called optimum asymptotic multiuser efficiency of a CDMA system with a spreading matrix $\boldsymbol{C}$ with i.i.d elements tends to 1 if the loading factor $\beta = K/L$ is kept equal to an arbitrary constant when K tends to infinity. This implies that for such a system, the spreading matrix $\boldsymbol{C}$ is uniquely detectable in the large system limit. However, the conditions in [31] are not satisfied when the spreading matrix is chosen based on the ACF property, for example, when the matrix is sparse. In the next theorem, we prove that in the large system limit, choosing random sparse matrices based on the ACF property, will result in uniquely detectable matrices with probability tending to 1.

Theorem 1: Let A be an $L \times K$ binary random matrix whose elements have Bernoulli (p) distribution [If z has a Bernoulli (p) distribution, its value is 1 with probability p and 0 with probability $1 - p$.] such that the average number of 1's in each row, that is, Kp, satisfies the following equation

$$Kp = K^{(1/\alpha)} \tag{45}$$

where α is a finite real number greater than 1. If K and L tend to infinity such that the loading factor is kept equal to an arbitrary constant, then A is uniquely detectable with probability approaching 1.

Proof: See the Appendix.

Theorem 1: Implies that if we select an $L \times K$ random matrix whose elements are in {0, 1} and the average number of non-zero elements in each row is equal to $K^{1/\alpha}$ for some $\alpha > 1$, it is uniquely detectable with probability 1 in the large system limit. Moreover, based on the ACF property such a matrix has no cycles of length shorter than $\alpha/4$.

In a system with finite length, choosing the value of α is very important. It can be seen that in the finite length systems, there is in fact an optimum value for α. In other words, there is an optimum value for the average number of non-zero elements in each row. In simulation results section, we show this fact for the first proposed receiver.

In the finite length systems as opposed to large system limit, we cannot guarantee that there is no cycles shorter than a given length and this can degrade the performance. Therefore, for the considered OCDMA system, we investigate the existences of graphs for which we can guarantee that there are no cycles of length 4. To do so, we need sparse codes that have at most one chip interference with each other. A natural choice of codes that satisfy this condition are OOC with low weight and their shifted versions when the maximum cross-correlation between codes is set to 1. We abbreviate such codes by shifted optical orthogonal codes (SOOC).

In general, an (L, w, λ_a, λ_c) OOC is a family of (0, 1) sequences of length L and weight w with maximum auto- and cross-correlation λ_a and λ_c, respectively. These codes are designed for asynchronous OCDMA systems. For synchronous OCDMA systems, the OOC codes and their cyclically shifted versions can be used. This makes it possible to significantly increase the number of codes compared with the asynchronous case. We use OOC's with small values for w (around 4 in our case) and $\lambda_c = 1$. In such a case, the maximum number of codes is obtained by Johnson's bound as follows [32]

$$N_c < \left\lfloor \frac{L}{w} \left\lfloor \frac{L-1}{w-1} \right\rfloor \right\rfloor \tag{46}$$

In the simulation results section, we compare the performance of the random codes with that of SOOC codes.

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6 Simulation results

In this section, simulation results are presented to demonstrate the performance of the proposed receivers in optical fibre channels. First, the performance of Algorithm 1 is investigated for random codes and SOOC codes in the noiseless channel. A noiseless channel can be easily obtained by letting $\lambda_n = 0$ and $\lambda_x \gg 1$.

Fig. 5 shows the BER performance of Algorithm 1 against the number of iterations using random codes of weight 5 for a system with $L = 64$ chips and with $K = 80$, 92 users in the noiseless channel. To do so, we have randomly generated the spreading matrix while guaranteeing to have exactly 5 ones on each column of the matrix. From Fig. 5, it is observed that a few number of iterations is enough to obtain the best performance of Algorithm 1. To achieve BER of 10^{-3}, the number of users is limited to 80, which is still in the overloaded region; however, we show that SOOC codes can accommodate a larger number of users to achieve the same BER.

In Fig. 6, we plot the BER performance of Algorithm 1 against the number of users for the random codes with different weights in the overloaded region. Moreover, the

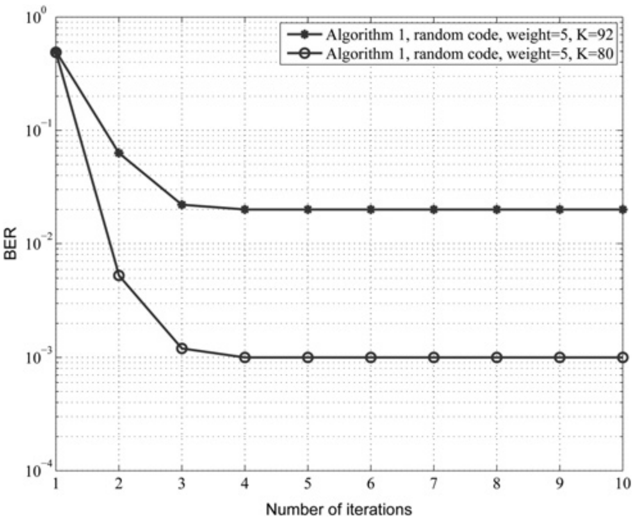

Fig. 5 *BER results of Algorithm 1 against the number of iterations ($L = 64$, $\lambda_n = 0$ and $\lambda_x \gg 1$)*

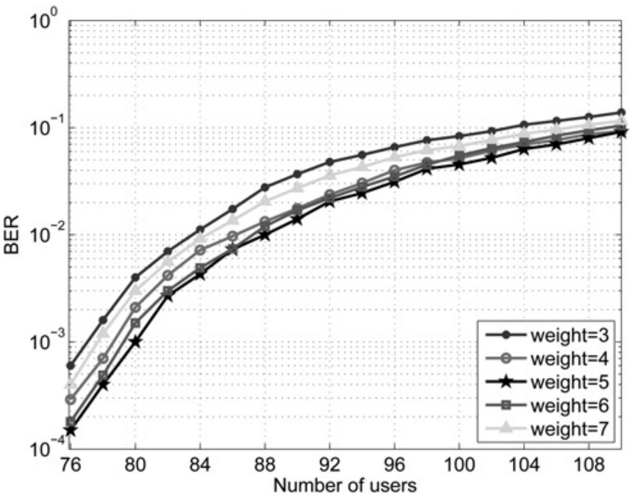

Fig. 6 *BER performance of Algorithm 1 for random codes ($L = 64$, $\lambda_n = 0$ and $\lambda_x \gg 1$)*

number of iteration is set 10 to obtain the best performance. As can be seen, there is an optimum value for the row weight of the spreading matrix, which is 5 in this case. The existence of this optimum weight is because of two sources of error in Algorithm 1 in the overloaded region. The first source of error is the cycles in the factor graph representation of the system [25] and the second one is the existence of multiple user data vectors for a given spread vector. The probability of having cycle increases when the weight increases. However, the probability of the second source of error decreases when the weight increases. Therefore there is an optimum weight that results in the best BER. Note that this optimum weight depends on the code length. In fact, for each code length L, there is an optimum weight that minimises the probability of error. Our simulation shows that the optimum weight for random codes with length 64 is 4.

Fig. 7 shows the performance of Algorithm 1 for the SOOC codes. In this figure, the same parameters as in Fig. 5 are used. We have also included the best random code from the previous figure. It can be observed from Fig. 7 that SOOC has a better performance than the random codes in terms of number of supported users. It is notable that the optimum value of the weight of codes for SOOC codes is 4. We also compare the performance of Algorithm 1 with that of conventional correlator and a new interference cancellation method presented in [33]. It can be observed that Algorithm 1 has a performance much better than the two other methods.

In the remaining parts of the simulation results, we investigate the performance of the proposed receivers in a real optical channel, that is, non-zero λ_x and finite λ_n and consider the effect of different system parameters, namely, number of iterations, number of chips, number of users and average photoelectron count on the BER performance of the three proposed algorithms. In the simulations, SOOC codes with weight 4 are used because of their superior performance as shown in Fig. 7.

In Fig. 8, we have investigated the BER performance of the proposed receivers against the number of iterations for different number of users. We have set $L = 128$, $\lambda_n = 0.1$ and $\lambda_x = 5$. Fig. 8 shows that the three proposed algorithms converge in a few number of iterations.

In Fig. 9, we have investigated the BER performance of the proposed methods against the number of chips for 20 users. We set $\lambda_n = 0.1$, $\lambda_x = 5$ and T the number of iterations to 10. From this figure, we see that the performance of Algorithm

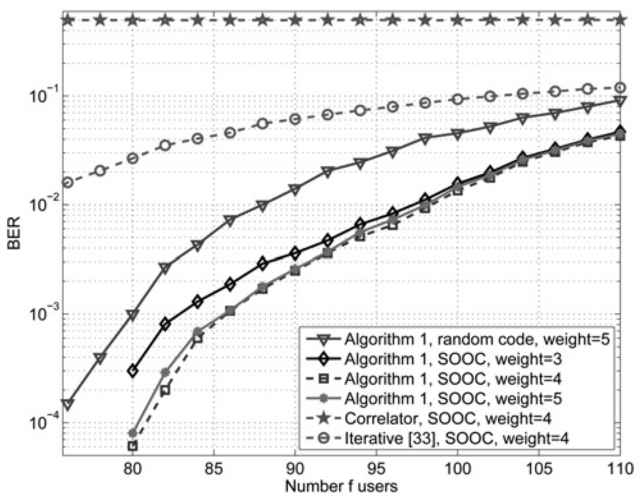

Fig. 7 *BER performance of Algorithm 1 for SOOC ($L = 64$, $\lambda_n = 0$ and $\lambda_x \gg 1$)*

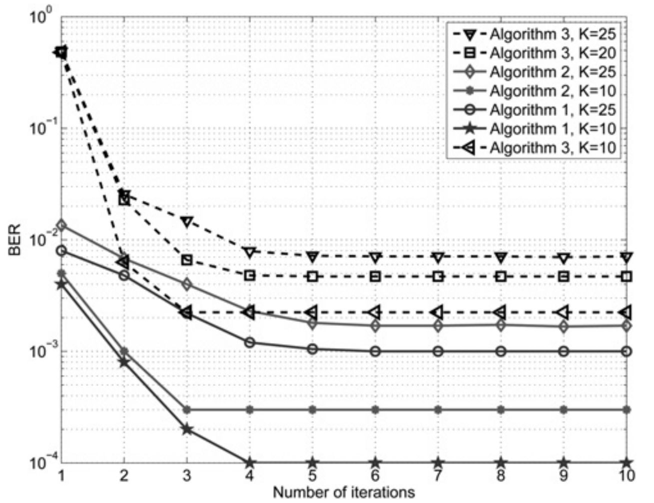

Fig. 8 *BER results of the proposed receivers against the number of iterations for various number of users (L = 128, $\lambda_x = 5$ and $\lambda_n = 0.1$)*

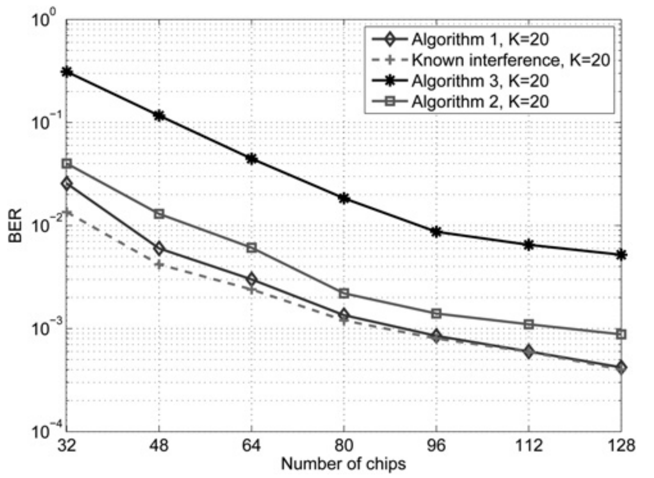

Fig. 9 *Performance comparison of the proposed receivers against the number of chips ($\lambda_x = 5$, $\lambda_n = 0.1$ and K = 20)*

1 is very close to the known interference lower bound for large number of chips. We also see that the performance of Algorithm 2 is pretty close to Algorithm 1 and the performance of Algorithm 3 is considerably worst than Algorithm 2. This is because of the approximation used in Algorithm 3 which results in (37) and (38). On the positive side, the complexity of this algorithm is less than that of Algorithm 2. Therefore the proposed BP-based receivers present different complexity/performance specifications.

Fig. 10 investigates the BER performance for the proposed receivers against the number of users when L (the number of chips) is set to 128. In this figure we set $\lambda_n = 0.1$ and $\lambda_x = 5$. As it can be seen from this figure, the performance of the proposed receivers is in similar order as Fig. 9.

In Fig. 11, we have investigated the BER performance against the average photo-electron count in each chip. We have set $L = 128$, $K = 30$, $\lambda_{\rm n} = 0.1$ and the number of iteration to 10. As Fig. 11 shows, the performance of Algorithm 1 is very close to the known interference lower bound and Algorithms 2 and 3 have worst performance than the known interference lower bound as predicted. By increasing the average photoelectron count which can be achieved by increasing the transmitted power, BER improves.

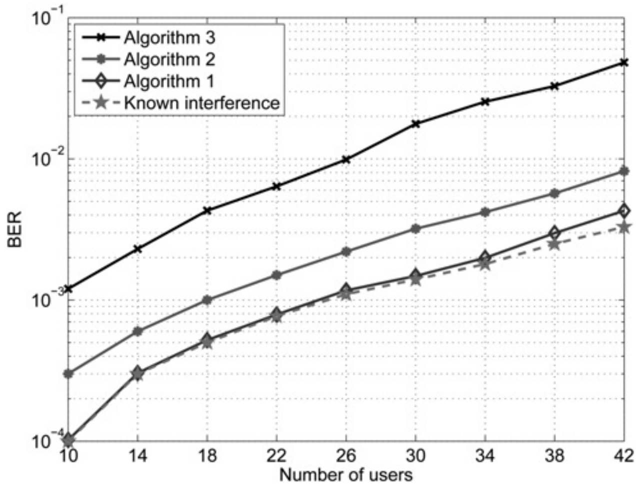

Fig. 10 *Performance comparison of the proposed methods against the number of users ($\lambda_n = 0.1$, $\lambda_x = 5$ and L = 128)*

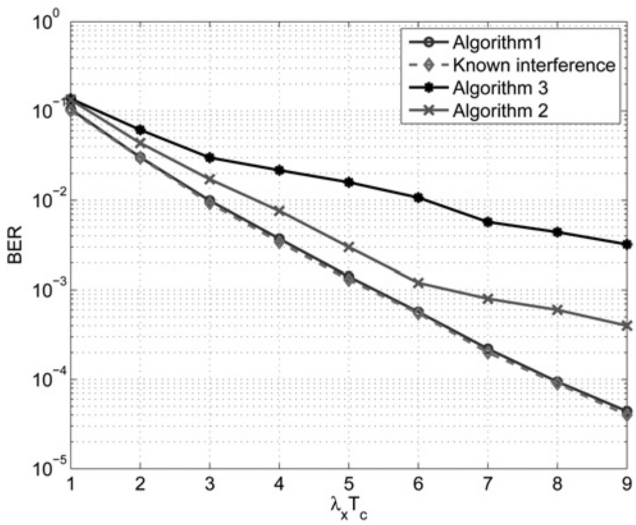

Fig. 11 *BER performance of the proposed receivers against the average of photo-electron count in each chip (λ_x) for 30 users ($\lambda_n = 0.1$ and L = 128)*

7 Conclusions

In this paper, three multiuser receivers for OCDMA systems based on the BP algorithm were proposed. The first proposed receiver was shown to exhibit a performance close to the known interference lower bound. The second receiver was obtained by applying some approximations to the first receiver to make it considerably simpler at the expense of a slight degradation in performance. It was shown that the third receiver is in fact identical to the multistage detector already proposed for OCDMA systems. We then investigated the spreading codes that can perform well with the proposed receivers. Consequently, we looked for sparse spreading matrices that are also uniquely detectable. We proved the existence of such matrices with probability approaching one in the large system limit. Nevertheless, systems with random sparse spreading matrices with finite dimensions can also exhibit a pretty good performance. For spreading codes of finite length, we also investigated the performance of SOOC codes when the maximum cross-correlation between codes is set to 1. In this case,

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there is no cycle with length 4 in the factor graph representation of the corresponding spreading matrix. It was shown that the proposed SOOC codes outperform random codes.

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10 Appendix

10.1 Proof of Theorem 1

Suppose that $P(B)$ is the probability that $\boldsymbol{A}$ is a uniquely detectable matrix. $\boldsymbol{A}$ is not uniquely detectable matrix if there is at least one $\boldsymbol{x} \in \{0,\ 1,\ -1\}^K$, $x \neq \boldsymbol{0}_{\{K\times 1\}}$ such that $\boldsymbol{A}x = 0_{\{L\times 1\}}$. Therefore by using the union bound we can write

$$\begin{aligned} P(B) &= 1 - P(\bar{B}) \\ &\geq 1 - \sum_{x \in \{0,1,-1\}^K, \boldsymbol{x} \neq 0_{\{K\times 1\}}} P\Big(\boldsymbol{A}x = 0_{\{L\times 1\}}\Big) \end{aligned} \tag{47}$$

It can be proven that $P(\boldsymbol{A}x = \boldsymbol{0}_{\{L\times 1\}})$ is the same for all $\boldsymbol{x}$'s with the same weight and the same number of 1's. [By the weight of a vector, we mean the number of non-zero elements in it.] Therefore the summation in (47) can be changed into two summations and we have

$$\begin{aligned} P(\bar{B}) \leq & \sum_{k=1}^{K} \binom{K}{k} \\ & \times \sum_{l=0}^{k} \binom{k}{l} P\Big(\boldsymbol{A}x = \boldsymbol{0}_{\{L\times 1\}} | w_{\boldsymbol{x}} = k,\ l_{\boldsymbol{x}} = l\Big) \end{aligned} \tag{48}$$

where w_x is the weight of the vector x and l_x is the number of 1's in x. The elements of each row of A are independent. Therefore $P(\boldsymbol{A}\boldsymbol{x} = 0_{\{L\times 1\}}/w_{\boldsymbol{x}} = k,\ l_{\boldsymbol{x}} = l)$ in (48) can be written as

$$\begin{aligned} & P\Big(\boldsymbol{A}\boldsymbol{x} = 0_{\{L\times 1\}} | w_{\boldsymbol{x}} = k,\ l_{\boldsymbol{x}} = l\Big) \\ &= \left[P\left(\sum_{j=1}^{K} \boldsymbol{A}_{i,j} x_j = 0 | w_{\boldsymbol{x}} = k,\ l_{\boldsymbol{x}} = l \right) \right]^L \\ &= \left[\sum_{r=0}^{\min(l,\, k-l)} \binom{k-l}{r} \binom{l}{r} p^{2r} (1-p)^{k-2r} \right]^L \end{aligned} \tag{49}$$

Now we try to find an upper bound for (49). The maximum value of (49) is obtained when $l = [k/2]$. We only obtain this upper bound when k is even. For odd k, the upper bound can be obtained in a similar way. For an even k, we can write

$$\sum_{r=0}^{\min(l,\,k-l)} \binom{k-l}{r}\binom{l}{r} p^{2r}(1-p)^{k-2r} \le \max_{r'\in\{0,\dots,k/2\}} \binom{k/2}{r'} p^{r'}(1-p)^{k/2-r'} \times \sum_{r=0}^{k/2}\binom{k/2}{r}p^{r}(1-p)^{k/2-r} = \max_{r\in\{0,\dots,k/2\}}\binom{k/2}{r}p^{r}(1-p)^{k/2-r} \tag{50}$$

It can be easily proven that the maximisation of (50) happens when

$$r = \left[\frac{kp}{2}\right] \tag{51}$$

Thus using (48)–(51), we have

$$P(\bar{B}) \le \sum_{k=1}^{K}\binom{K}{k}2^{k}\left[\binom{k/2}{[kp/2]}p^{[kp/2]}(1-p)^{k/2-[kp/2]}\right]^{L} \tag{52}$$

Let k_0 be an integer number such that if $k \le k_0$, then $[kp/2] = 0$. Hence (52) can be written as

$$P(\bar{B}) \le \sum_{k=1}^{k_0}\binom{K}{k}2^{k}(1-p)^{(kL/2)} + \sum_{k=k_0+1}^{K}\binom{K}{k}2^{k}\left[\binom{k/2}{[kp/2]}p^{[kp/2]}(1-p)^{k/2-[kp/2]}\right]^{L} \tag{53}$$

Let z be defined as

$$z = k/K \tag{54}$$

We first calculate the first summation in (53) in the large system limit as

$$\lim_{K,L\to\infty}\sum_{k=1}^{k_0}\binom{K}{k}2^{k}(1-p)^{kL/2} = \lim_{K,L\to\infty}\int_{1/K}^{k_0/K} 2^{KH(z)+Kz+(1/2)KzL\,\log_2(1-p)}K\,\mathrm{d}z \tag{55}$$

In (55), we used the following asymptotic approximation

$$\binom{K}{kz} \simeq 2^{KH(z)} \tag{56}$$

where $H(t) = -t\,\log_2 t - (1-t)\,\log_2(1-t)$. Since based on (45), p tends to zero in large system limit, we can use the following approximation in the large system limit

$$\log_2(1-p) \simeq -p\,\log_2(e) = -K^{(1/\alpha)-1}\,\log_2(e) \tag{57}$$

Using (57), the right-hand side of (55) can be written as

$$\lim_{K,L\to\infty}\int_{1/K}^{k_0/K} 2^{K\left(H(z)+z-(1/2\beta)K^{(1/\alpha)}z\,\log_2(e)\right)}K\,\mathrm{d}z \tag{58}$$

For finite values α and β, it can be easily proven that for $z \in [1/K,\, k_0/K]$

$$H(z) + z - \frac{1}{2\beta}K^{(1/\alpha)}z\,\log_2(e) < 0 \tag{59}$$

and therefore the limit in (58) is equal to zero.

For the second summation in (53) in the large system limit, we have

$$\lim_{K,L\to\infty}\sum_{k=k_0+1}^{K}\binom{K}{k}2^{k}\left[\binom{k/2}{[kp/2]}p^{[(kp/2)]}(1-p)^{k/2-[(kp/2)]}\right]^{L} \le \lim_{K,L\to\infty}\sum_{k=k_0+1}^{K}\binom{K}{k}2^{k}\left[\sqrt{\pi kp(1-p)}\right]^{-L} = \lim_{K,L\to\infty}\int_{(k_0+1)/K}^{1} 2^{K\left[H(z)+z-(1/2\beta)\left(\log_2\left(\pi zK^{(1/\alpha)}\right)\right)\right]}K\,\mathrm{d}z \tag{60}$$

where we used the following inequality

$$\binom{K}{Kz} \le 2^{KH(z)}\Big/\left(\sqrt{2\pi Kz(1-z)}\right) \tag{61}$$

For finite values of α and β, it is easy to prove that for $z \in [((k_0+1)/K),\, 1]$

$$H(z) + z - \frac{1}{2\beta}\log_2\left(\pi zK^{(1/\alpha)}\right) < 0 \tag{62}$$

and therefore the right-hand side of equality in (60) is equal to zero. Since both terms in the right-hand side of (53) tend to zero in the large system limit, we conclude that

$$\lim_{K,L\to\infty} P(\bar{B}) = 0 \tag{63}$$

and consequently

$$\lim_{K,L\to\infty} P(B) = 1 \tag{64}$$

This proves the theorem.

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