ANALYSIS AND RECOVERY OF MULTIDIMENSIONAL SIGNALS FROM IRREGULAR SAMPLES USING NON-LINEAR AND ITERATIVE TECHNIQUES

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Abstract - A recurring problem in signal processing is reconstruction based upon imperfect information. In this paper we offer two methods. The first one is a simple non-linear method, based upon spectral analysis, to reconstruct two-dimensional (2-D) band-limited signals from nonuniform samples. The second method is an iterative technique; by using an intelligent initial condition, we get faster convergence than a simple successive approximation. The feasibility of this procedure is demonstrated through examples of color image restoration when some of the pixels are lost.

I. Introduction

Irregular sampling is more the rule than the exception for many types of band-limited signals, particularly where the samples deviate from uniform positions only by a finite amount. One example is the case of lost pixels in a recording channel or a fading transmission medium[1]. The remaining pixels form a set of nonuniform samples.

In the next section, we analyse the spectrum of two-dimensional (2-D) irregularly sampled signals, leading to the non-linear signal recovery method. In section III the non-linear signal recovery and an iterative reconstruction technique are discussed. Experimental results are given in section VI for 2-D signals – comparing the non-linear technique, in terms of signal-to-noise ratios, to the iterative technique and other standard techniques.

II. Spectral Analysis of Irregular Samples

In this section, we would like to analyze the spectrum of nonuniform samples that deviate from uniform positions by a finite amount. The derivation of the spectrum of nonuniform samples leads to a practical method to recover a 2-D signal from the nonuniform samples.

We model the set of 2-D nonuniform samples by

$$f_s(x,y) = \sum_m\sum_n f(x_{nm},y_{nm})\,\delta(x-x_{nm}\,,\ y-y_{nm}), \tag{1}$$

where $f(x,y)$ is a band-limited 2-D signal. The irregular set $\{x_{nm}, y_{nm}\}$ is assumed to deviate from the uniform set $\{nT_1, mT_2\}$ as seen in Figure 1.

To analyze the spectrum of (1), we first define the *comb* function as

$$f_p(x,y) = \sum_m\sum_n \delta(x-x_{nm}\,,\ y-y_{nm}). \tag{2}$$

Now, define the following functions

$$\begin{cases} g_1(x,y) = x - nT_1 - \theta_1(x,y) \\ g_2(x,y) = y - mT_2 - \theta_2(x,y), \end{cases} \tag{3}$$

such that $g_1(x_{nm},y_{nm}) = g_2(x_{nm},y_{nm}) = 0$ for any n and m. $\theta_1(x,y)$ and $\theta_2(x,y)$ are any two-dimensional functions such that

$$\begin{cases} \theta_1(x_{nm},y_{nm}) = x_{nm} - nT_1 \\ \theta_2(y_{nm},y_{nm}) = y_{nm} - mT_2, \end{cases} \tag{4}$$

Equation (4) can be interpreted as deviation of nonuniform points from uniform positions.

Now, we have the following identity:

$$\delta(x-x_{nm},y-y_{nm}) = \delta[g_1(x,y),g_2(x,y)]\,|J|, \tag{5}$$

where J is the Jacobian defined by

$$J = \begin{vmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_2}{\partial x} \\ \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_2}{\partial y} \end{vmatrix} = \frac{\partial g_1}{\partial x}\frac{\partial g_2}{\partial y} - \frac{\partial g_1}{\partial y}\frac{\partial g_2}{\partial x}. \tag{6}$$

Now by using (3), (4), (5) and (6), the *comb* function (2) can be written as

$$f_p(x,y) = \sum_m\sum_n |J|\,\delta(x-nT_1-\theta_1(x,y), y-mT_2-\theta_2(x,y)). \tag{7}$$

Using the notation $x_1 = x-\theta_1(x,y)$ and $y_1 = y-\theta_2(x,y)$, and using Fourier series expansion, we get

$$f_p = \frac{|J|}{T_1T_2}\{1+2\sum_{i=1}^{\infty}\cos\frac{2\pi i}{T_1}[x-\theta_1]\}\{1+2\sum_{l=1}^{\infty}\cos\frac{2\pi l}{T_2}[y-\theta_2]\}. \tag{8}$$

Equation (8) is equivalent to the sum of phase modulated signals in two-dimensions. Therefore, the Fourier spectrum of $f_p(x,y)$ consists of a low-pass component $\dfrac{|J_1|}{T_1T_2}$ and band-pass or high-pass components around carrier frequencies $(\dfrac{2\pi n}{T_1}, \dfrac{2\pi m}{T_2})$.

The nonuniform samples (1) are derived by multiplying (8) by $f(x,y)$, i.e.,

$$f_s(x,y) = f(x,y)f_p(x,y) = f(x,y)\,\frac{|J|}{T_1T_2}$$

$$\{1+2\sum_{i=1}^{\infty}\cos\frac{2\pi i}{T_1}[x-\theta_1(x,y)]\}\{1+2\sum_{l=1}^{\infty}\cos\frac{2\pi l}{T_2}[y-\theta_2(x,y)]\}. \tag{9}$$

This spectral analysis leads to a practical reconstruction method, which is our next topic of discussion.

III. Practical Reconstruction Techniques

In this section we shall discuss two practical reconstruction methods for image recovery from a set of nonuniform samples, namely a non-linear method and an iterative technique.

3.1. A Non-linear Recovery Method

Spectral analysis reveals that the information about the original signal lies in the low-pass component of the ideal nonuniform samples (9); i.e., low-pass filtering the nonuniform samples (9), we get

$$P[f_s(x,y)] \approx f(x,y)\,\frac{|J|}{T_1T_2}, \tag{10}$$

where P is a low-pass operator. The Jacobian in (10) can be evaluated by low-pass filtering the *comb* function (8). The result is

$$P[f_p(x,y)] \approx \frac{|J|}{T_1T_2}. \tag{11}$$

The approximation in (10) and (11) is due to the spectral overlap of phase modulated components. A division of (10) by (11) yields

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a good approximation of the original signal $f(x,y)$. The division is possible if the Jacobian function is not equal to zero. This condition puts a limitation on the sampling point deviations. A sufficient condition is that the maximum values of $\theta_1(x,y)$ and $\theta_2(x,y)$ in (4)- which are always greater than or equal to the nonuniform deviations in (3)- be less than $\frac{T_1}{2\pi}$ and $\frac{T_2}{2\pi}$, respectively.

3. An Iterative Recovery Method

An iterative method has been developed to recover the nonuniform samples. The iteration is as follows:

$f(x,y) = \lim_{n\to\infty} f_n(x,y)$,

where

$$f_{n+1}(x,y) = \frac{k_1}{k_2}[PS\, f(x,y) - PS\, f_n(x,y)] + f_n(x,y), \qquad (10)$$

where P and S are, respectively, the low-pass filtering and the nonuniform sampling process for a 2-D signal. Note that the above iteration is not identical to the iterative methods discussed in [2] and [3]. Reference [2] uses smeared pulses as opposed to ideal impulsive samples, and reference [3] uses the concept of Projection Onto Convex Sets (POCS) for the ideal impulsive samples and the iteration is based on sample by sample (for more details see[6]). The above iterative technique is an extension to the one dimensional case[4]. The proof of the convergence is, however, more involved and is given in Appendix A.

IV. Simulation Results

To simulate the nonuniform samples, we consider the important special case where in a uniform grid of points, some of the pixels are randomly lost. The remaining set of pixels form a set of nonuniform samples. In the following, we simulate the non-linear method, the iterative method, and compare it to other standard techniques such as low-pass filtering and mean filtering (Median filtering was also simulated but will not be discussed here due to its inferior performance compared to mean filtering).

The Non-linear Method

The non-linear method has been tested on a Sun Station for a 256×256 band-limited color image as shown in Fig. 2, sampled at twice the Nyquist rate. To create nonuniform samples of the image, a random array is generated from IMSL library. After quantizing the random samples, we get a 256×256 array of, for example, about 80% 1's and 20% 0's. This array shows a 20% pixel loss and determines the nonuniform positions $f_p(x,y)$, and therefore, by low-pass filtering this array, one gets the Jacobian $|J|$, Equation (8). After multiplying this array $(f_p(x,y))$ by the image, we get the nonuniform samples $-f_s(x,y)$ – of the image at the Nyquist rate, Equation (9). Low-pass filtering the nonuniform samples (9), we get Fig. 3, which has significant distortion. The division of the low-pass image represented in Fig. 3 by Jacobian $|J|$ yields the image shown in Fig. 4. As Fig. 4 reveals, an impressive improvement of 18 dB over simple low-pass filtering. A simulation of a 2-D linear interpolation (Mean Filtering) is also performed for comparison. In mean filtering, we average an array of 3X3 pixels to get an estimate of a lost pixel at the center of the array. We find that the non-linear technique is about 2.4 dB better than mean filtering. The non-linear technique becomes much better than the mean filtering when the sampling rate is increased, or alternatively, if the pixel loss is lower than 20%.

Although significant improvement is observed, some white spikes appear in the picture. These white spikes are due to the fact that at higher sample losses, the probability of loss of blocks of pixels increases. The output of the low-pass filter about these blocks is very close to zero. Therefore, a division by zero occurs. To alleviate this problem, we use another hard-limiter at the end of the divider followed by a low-pass filter. This additional process tends to smooth the undesirable spikes.

The Iterative Method

For the iterative method, if the initial guess $f_0(x,y)$ is properly chosen, the speed of convergence can be increased. In our simulation, we choose the mean filtered image as the initial estimate for $f_0(x,y)$. Figure 5 is the degraded image after 40% pixel loss. Figure 6 is the corresponding recovered images after 15 iterations. The results are surprisingly indistinguishable with the original image (Figure 2). As expected with more sample losses, more iterations are needed to get the same quality of image.

V. Conclusion

In this paper we presented a method to describe nonuniform samples of a 2-D signal in the frequency domain. The spectral analysis suggests a simple non-linear method to reconstruct the signal from its nonuniform samples. The same procedure can be generalized to n-dimensional signals. Since the non-linear method requires a division, we have to make sure that the denominator is not zero. This requirement puts a constraint on the position of nonuniform points. The error decreases as the set of irregular samples gets closer to the set of uniform sampling set. Or in terms of pixel losses, significant improvement is observed with the decrease of the percentage of pixel losses. In comparison to mean filtering, the nonlinear technique is always better at random pixel losses. In case a block of adjacent pixels is lost, mean filtering may outperform the nonlinear technique.

We also developed a theoretical background to to recover images from lost samples using iterative techniques. This technique is based on ideal impulsive nonuniform samples and the iteration is on the whole image as opposed to pixel by pixel. In comparison, the non-linear method is easy to implement and performs quite well, particularly, when the percentage of the lost samples is low. However, for a high quality reconstructed image or a high percentage of lost pixels, the iterative method is preferred.

Appendix A

Here we prove that if the sampling set x_{xm}, y_{nm} is a stable set, the iteration given in (10) will converge to a stable point, which is equal to the original image.

$$f_{k+1}(x,y) = \lambda PS\, f(x,y) + (P - \lambda PS) f_k(x,y) \qquad \text{(A-1)}$$

where λ , f(x,y) and f_k(x,y) are a convergence constant (relaxation parameter), the original finite energy signal and the k^{th} iteration, respectively. P and S are, respectively, the 2-D band-limiting and ideal nonuniform sampling operators. $PSf(x,y)$ in (A-1) is the low-pass filtered version of the ideal nonuniform samples, which is known. In the following, we prove that there exists a range for λ for which the process will converge to a stable point, which is equal to the desired signal f, i.e.,

$\lim_{k\to\infty} f_k(x,y) \to f(x,y)$

Proof:

The iteration will converge : $\{\lim_{k\to\infty} f_k \to f(x,y)\}$ if $P - \lambda PS$ is a contraction. The operator $P - \lambda PS$ is a contraction [4] if

$$|| f^{(k+1)}(x,y) || < || f^{(k)}(x,y) || \quad \text{for all } k \qquad \text{(A-2)}$$

where $\|f(x,y)\|$ is the L^2 norm of $f(x,y)$ and $f^{(k)}(x,y) = f_k(x,y) - f_{k-1}(x,y)$.

or from (A-1), we have

$$|| P(f^{(k)}(x,y)) - \lambda PS(f^{(k)}(x,y)) || \leq r\, || f^{(k)}(x,y) || \qquad \text{(A-3)}$$

where $0 \leq r < 1$.

So if (A-3) is satisfied, the theorem is proved. We now show that there is a λ such that (A-3) is true. The left hand side of (A-3) can be written as

$$|| P(f^{(k)}(x,y)) - \lambda PS(f^{(k)}(x,y)) ||^2 = || f^{(k)}(x,y) ||^2 + \qquad \text{(A-4)}$$

$$\lambda^2 ||PS(f^{(k)}(x,y))||^2 - 2\lambda \int\int P(f^{(k)}(x,y))\, PS(f^{(k)}(x,y))\, dx dy$$

Now, we show that there is a positive real k_1 and k_2 such that

$$\int\int P(f^{(k)}(x,y))\ PS(f^{(k)}(x,y))\,dxdy \geq k_1 \,||\, f^{(k)}(x,y) ||^2 \qquad \text{(A-5)}$$

and

$$||\, PS(f^{(k)}(x,y)) \,||^2 \leq k_2 \,||\, f^{(k)}(x,y) \,||^2 \qquad \text{(A-6)}$$

To prove (A-5), we first note that

$$\int\int P(f^{(k)})\ PS(f^{(k)})\,dxdy = \int\int (f^{(k)})\ S(f^{(k)})\,dxdy, \qquad \text{(A-7)}$$

where $f^k = f^k(x,y)$. since the operator P is self-adjoint (i.e., $\int (Px)\,y\,dt = \int x(Py)\,dt$), $P^2 = P$, and $Px_k = x_k$.

Now, equation (A-7) can be written as

$$\int\int (f^{(k)}(x,y))\ S(f^{(k)}(x,y))\,dxdy = \sum_n\sum_m [f^{(k)}(x_{nm}, y_{nm})]^2 \qquad \text{(A-8)}$$

Since the points $\{ x_{nm}, y_{nm} \}$ are assumed to be a stable sampling set, from Nikoliskii inequality[5] we have for all signals f that are band-limited

$$A \leq \frac{\sum_n\sum_m f^2(x_{nm}, y_{nm})}{||\, f(x,y) \,||^2} \leq B \qquad \text{(A-9)}$$

where A and B are positive constants, depending only on the bandwidth and $\{x_{nm}, y_{nm}\}$.

From (A-7),(A-8) and (A-9), we conclude

$$\int\int P\ (f^{(k)}(x,y))\ PS\ (f^{(k)}(x,y))\,dxdy \geq A \,||\, f^{(k)}(x,y) ||^2 \qquad \text{(A-10)}$$

Therefore, (A-5) is proved by setting $k_1 = A$.

To prove (A-6), we can write

$$|PS(f^{(k)})|^2 = \int\int PS(f^{(k)}) \sum_n\sum_m (f^{(k)}(x_{nm}, y_{nm})\,\delta(x - x_{nm}, y - y_{nm})\,dxdy$$

$$= \sum_n\sum_m \int\int [PS(f^{(k)})](f^{(k)}(x_{nm}, y_{nm})\ \delta(x - x_{nm}, y - y_{nm})\,dxdy$$

$$= \sum_n\sum_m [PS(f^{(k)}(x,y))](x_{nm}, y_{nm})(f^{(k)}(x_{nm}, y_{nm})$$

$$\leq [\ \sum_n\sum_m |\, f^{(k)}(x_{nm}, y_{nm}) \,|^2]^{1/2} [\sum_n\sum_m |\, [PS(f^{(k)})](x_{nm}, y_{nm}) \,|^2]^{1/2}$$

$$\leq B\, ||f^{(k)}|| \,.\, ||PS(f^{(k)}|| \qquad \text{(A-11)}$$

Hence, from (A-9), we have

$$||PS(f^{(k)}(x,y))\,|| \leq B\, ||f^{(k)}(x,y)|| \qquad \text{(A-12)}$$

Therefore, hypothesis (A-6) is proved. From (A-4),(A-5) and (A-6), we get

$$||\, P\ (f^{(k)}(x,y)) - \lambda PS\ (f^{(k)}(x,y)) \,||^2 \leq r \,||\, f^{(k)}(x,y) ||^2$$

where $r = 1 + \lambda^2 k_2 - 2\lambda k_1$

In order to satisfy (A-3), r has to be in the range of $0 \leq r < 1$, i.e., given a particular k_1 and k_2 , λ has to be in the following region of convergence for the iterative relationship given in (A-1)

$$\begin{cases} 0 < \lambda < \dfrac{2k_1}{k_2} \\ k_1^2 \leq k_2 \end{cases} \qquad \text{(A-13)}$$

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References

[1] F. Marvasti, "Spectrum of Nonuniform Samples " Electronic Letters vol.20, Oct. 1984.

[2] K. D Sauer, J. P. Alleback, " Iterative Reconstruction of Band-limited Images from Nonuniformly Spaced Samples," IEEE Trans. on Circuit and Systems, Vol. CAS-34, No.12, Dec. 1987.

[3] S. Yeh and H. Stark, "Iterative and one-step reconstruction from nonuniform samples by convex projections," J. of Opt. Soc. of America A., vol. 7, no. 3, pp. 491-499, March 1990.

[4] F. Marvasti, M. Analoui, and M. Gamshadzahi, "Recovery of Signals from Nonuniform Samples Using Iterative Methods," IEEE Trans. SP, vol. 39, no. 4, pp. 872-877, April 1991.

[5] Paul L. Butzer and Guido Hinsen, "Two-dimensional nonuniform sampling expansion- An iterative approach. I. Theory of Two-dimensional band-limited signals," Applicable Analysis, Vol. 32, pp. 53-67, Gordon and Breach, Science Publishers, Inc. 1989.

[6] C. Cenkar, H. Feichtinger, M. Hermann, "Iterative algorithms in irregular sampling: A first comparison of methods," in Conf. Proc. Phoenix Conference on Computers and Communications, Scottsdale, Arizona, March, 1991.

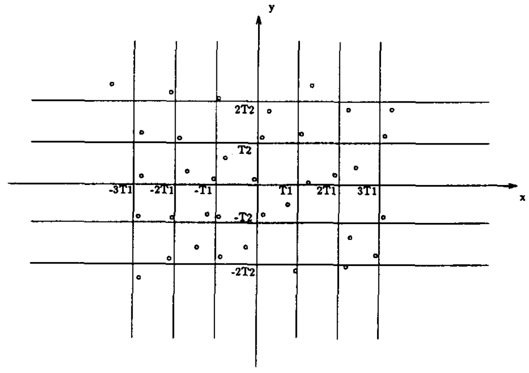

Figure 1

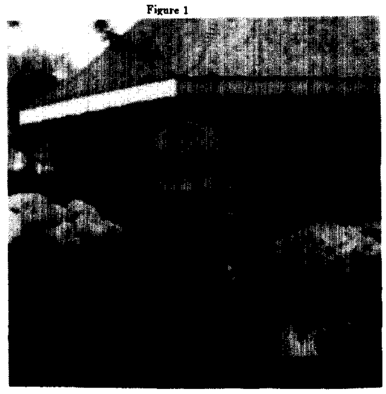

Fig.2 The original bandlimited image.

Fig.3 Low-Pass filtering(20% lost).

Fig.4 Division(20% lost).

Fig.5 40% degraded image.

Fig.6 15 iterations