Joint Pilot Power & Pattern Design for Compressive OFDM Channel Estimation

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***Abstract*—This letter investigates the deterministic design of pilot *power* and *pattern* for sparse channel estimation in OFDM systems based on minimizing the coherence of the DFT submatrix. It has been suggested that the pilot pattern forming a cyclic difference set (CDS) or almost difference set (ADS) is optimal. So, we proposed a deterministic procedure that jointly optimized for pattern and power of pilots as a solution. First, pilot patterns forming CDS/ADS were gathered through a search. Then, the power was numerically allocated to the different pilots from all the patterns. Finally, the pilot pattern and power pair leading to minimum coherence was selected from the available pairs. Simulation results demonstrated that the proposed scheme outperformed the existing methods in terms of efficiency.**

***Index Terms*—Coherence minimization, compressed sensing, OFDM channel estimation, pilot design, cyclic difference set, almost difference set.**

I. Introduction

ORTHOGONAL frequency division multiplexing (OFDM) is a popular communication system primarily because of its spectral efficiency and simple implementation. In OFDM systems, the frequency-selective wireless channel is divided into several flat fading sub-channels, resulting in a significant reduction in the transceiver complexity. However, channel estimation is an important issue in OFDM systems, since accurate information regarding the channel state can notably improve performance.

Recently, compressed sensing (CS) has been widely used to exploit the inherent sparsity of wireless channels to obtain a better estimate of the channel. This phenomenon is due to the sparse distribution of scatterers in space [1]. It is known that equi-distant pilots are optimal for least-square (LS) channel estimation techniques [2]. However, this is not true when sparse recovery algorithms are used. Also, restricted isometry property (RIP) is a criterion for optimization of the pilot pattern [3]. The intractability of checking for the RIP of a matrix motivated researchers to design a pilot pattern based on the *coherence* [4], [5]. In [5], discrete stochastic approximation is used to minimize the coherence. However, in [6], the coherence is replaced by the ℓ_2-norm of a vector whose entries are the correlations between different columns of the discrete Fourier transform (DFT) sub-matrix.

In this letter, a deterministic design of pilot *power* and *pattern* in OFDM systems, based on minimizing the coherence of the DFT sub-matrix, is investigated. We propose a deterministic procedure that jointly optimizes for the pattern and power of pilots. A cyclic difference set (CDS) and an almost difference set (ADS) are suggested for the pilot pattern.

The rest of the letter is organized as follows. In Section II, channel estimation in OFDM systems is formulated as a CS problem. Section III presents theoretical results and the proposed scheme. Simulation results are demonstrated in Section IV. Finally, Section V concludes the paper. The notations used are listed in Table I.

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TABLE I
Table of Notations

Symbol	Meaning
$(.)^T$	transpose
$(.)^H$	hermitian
$\text{diag}\{.\}$	diagonal Matrix
$\|.\|_2$	ℓ_2-norm
$\lfloor . \rfloor$	floor
$((.))_N$	modulo N
$\mathbb{Z}_N$	set of integers modulo N
$\mathbb{Z}$	set of integers
$\#\{.\}$	cardinality
$\text{var}(.)$	variance

II. System Description

In OFDM systems with a comb-type pilot arrangement where N_P of N sub-carriers are dedicated to transmission of pilots, ignoring inter-symbol interference and inter-carrier interference, the received data symbols $\mathbf{y} = [Y(1), Y(2), \cdots, Y(N)]^T$ can be formulated as:

$$\mathbf{y} = \mathbf{X}H + \eta \tag{1}$$

where $\mathbf{X} = \text{diag}\{X(1), X(2), \cdots, X(N)\}$ is a diagonal matrix whose elements are the transmitted data symbols, H denotes the channel frequency response (CFR), and η is the additive Gaussian noise.

Unlike conventional channel estimation methods that estimate the CFR at the pilot sub-carriers and then interpolate it at the other sub-carriers, compressive channel estimation techniques try to directly estimate the sparse channel impulse response (CIR). Therefore, channel estimation in OFDM systems can be stated as a CS problem as shown below:

$$\mathbf{y_P} = \mathbf{A}\mathbf{h} + \eta_\mathbf{P} \tag{2}$$

where $\mathbf{y_P} = [Y(p_1), Y(p_2), \cdots, Y(p_{N_P})]^T$ and $\eta_\mathbf{P} = [\eta(p_1), \eta(p_2), \cdots, \eta(p_{N_P})]^T$ are the received data and noise vector, at pilot locations: $\mathcal{P} = \{p_1, p_2, \cdots, p_{N_P}\} \subseteq \{1, \cdots, N\}$, respectively. Moreover, $\mathbf{A} = \mathbf{X}_P\mathbf{F}_P$ is the sensing matrix, $\mathbf{X}_P = \text{diag}\{X(p_1), X(p_2), \cdots, X(p_{N_P})\}$ and

$$\mathbf{F}_P = \begin{bmatrix} 1 & \omega^{p_1} & \cdots & \omega^{p_1(L-1)} \\ 1 & \omega^{p_2} & \cdots & \omega^{p_2(L-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{p_{N_P}} & \cdots & \omega^{p_{N_P}(L-1)} \end{bmatrix}_{N_P \times L}, \ \omega = e^{-j\frac{2\pi}{N}},$$

denotes the DFT sub-matrix with rows corresponding to pilot locations, and $\mathbf{h}_{L\times 1}$ represents the sparse CIR.

III. Pilot Power & Pattern Optimization

Recoverability is one of the main issues in CS theory. It implies whether an unknown signal with n entries can be recovered from measurements far fewer than n [7]. Since there is currently no polynomial-time algorithm to check the RIP of a matrix [8], the coherence of the sensing matrix is commonly used to guarantee the recovery. To achieve more recoverability, matrices with low coherence are desired.

A. Problem Statement

We start by minimizing the coherence of $\mathbf{A}$ to find optimal pilot design:

$$\mu(\mathbf{A}) \triangleq \max_{k\neq l} \frac{|\mathbf{a}_k^H \mathbf{a}_l|}{\|\mathbf{a}_k\|_2 \cdot \|\mathbf{a}_l\|_2}, \tag{3}$$

where $\mathbf{a}_k$ is the kth column of $\mathbf{A}$. One can show that:

$$\mu(\mathbf{A}) = \max_{1\leq l\leq L-1} \frac{\left|\sum_{i=1}^{N_P} |X(p_i)|^2 \omega^{p_i l}\right|}{\sum_{i=1}^{N_P} |X(p_i)|^2}. \tag{4}$$

Minimizing $\mu(\mathbf{A})$ is equivalent to minimizing:

$$g(\mathcal{P},\mathbf{t}) \triangleq \max_{1\leq l\leq L-1} h_l(\mathcal{P},\mathbf{t}), \tag{5}$$

where $h_l(\mathcal{P},\mathbf{t}) = \frac{\sum_{i=1}^{N_P}\sum_{j=1}^{N_P} t_i t_j \omega^{(p_i-p_j)l}}{\left(\sum_{i=1}^{N_P} t_i\right)^2}$ and $t_i \triangleq |X(p_i)|^2$ is the power of the ith pilot symbol. Note that:

$$g(\mathcal{P},\mathbf{t}) \geq \frac{1}{L-1}\sum_{l=1}^{L-1} h_l(\mathcal{P},\mathbf{t}), \tag{6}$$

and the equality holds when:

$$h_1(\mathcal{P},\mathbf{t}) = h_2(\mathcal{P},\mathbf{t}) = \cdots = h_{L-1}(\mathcal{P},\mathbf{t}). \tag{7}$$

Therefore, one can write $h_l(\mathcal{P},\mathbf{t})$ as follows:

$$h_l(\mathcal{P},\mathbf{t}) = \frac{\sum_{d=0}^{N-1}\left(\sum_{(i,j)\in S_d} t_i t_j\right)\omega^{ld}}{\left(\sum_{i=1}^{N_P} t_i\right)^2}, \tag{8}$$

where $S_d \triangleq \{(i,j) \mid ((p_i-p_j))_N = d\}$. Now, similar to [5], combining (7) and (8) results in[1]:

$$\sum_{(i,j)\in S_d} t_i t_j = C, \quad d = 1,2,\cdots,N-1, \tag{9}$$

where $C \triangleq \frac{1}{N-1}\sum_{(i,j)\in S} t_i t_j$ and $S \triangleq S_1 \cup S_2 \cup \cdots \cup S_{N-1} = \{(i,j), \quad 1\leq i,j\leq N_P, \quad i\neq j\}$. A solution is suggested using the definition of ADS and CDS [9], [10]:

[1]Note that the DFT of a signal which is equal everywhere along the time except at zero, is of the same form.

Definition 1: Let $\mathcal{P}$ be an N_P-subset of $\mathbb{Z}_N$ and $\lambda \triangleq \frac{N_P(N_P-1)}{N-1}$. The set $\mathcal{P}$ is a $(N,N_P,\lfloor\lambda\rfloor,K)$-ADS if the difference multiset $\mathcal{D} = \{a_d\}_{d=1}^{N-1}$ has K elements with the value $\lfloor\lambda\rfloor$ and $N-1-K$ elements with the value $\lfloor\lambda\rfloor+1$, where a_d is the number of pairs $(p_i,p_j)\in\mathcal{P}\times\mathcal{P}$, such that $((p_i-p_j))_N = d$ for $d=1,2,\cdots,N-1$. If $K=N-1$, then $\lambda\in\mathbb{Z}$ and $\mathcal{P}$ is a (N,N_P,λ)-CDS.

Now assume that $t_1 = t_2 = \cdots = t_{N_P}$, then (9) is reduced to:

$$\#\{S_d\} = \lambda = \frac{N_P(N_P-1)}{N-1}, \quad d = 1,2,\cdots,N-1 \tag{10}$$

which simply leads to CDS when λ is an integer. CDSs are well-known structures meeting the Welch bound [9], but they only exist for a few pairs of (N,N_P). There have been many previous attempts to nearly meet the Welch bound, resulting in the introduction of many classes of ADSs [10]–[12].

Equation (10) indicates that pilots with equal powers are optimal, if a CDS is available. This fact motivated us to take the power allocation into account when a CDS is unavailable. Furthermore, after considering (10), it is clear that an optimal solution may exist in view of the possibility of partitioning S into subsets with equal cardinality. Accordingly, we propose a backtracking algorithm that finds a near-optimal pilot pattern based on minimizing variance of the multiset of cardinalities $\{\#\{S_d\}\}_{d=1}^{N-1}$. Similarly, in the greedy algorithm presented in [4], the pilot pattern is updated so that a difference multiset with minimum variance is achieved in each iteration.

B. Proposed Scheme

For convenience, $\mathcal{D} = \{a_d\}$ is used to represent $\{\#\{S_d\}\}$, where a_d is defined in Definition 1. In Appendix A, it is shown that the difference multiset $\mathcal{D}$ of a set $\mathcal{P}$ has the minimum variance, if $\mathcal{P}$ is a $(N,N_P,\lfloor\lambda\rfloor,K)$-ADS, where $K \triangleq (N-1)(\lfloor\lambda\rfloor+1) - N_P(N_P-1)$. However, if $\lambda\in\mathbb{Z}$ then $\mathcal{P}$ is a (N,N_P,λ)-CDS. Therefore, finding pilot patterns forming either CDS or ADS is an essential part of the algorithm. After gathering pilot patterns forming CDS/ADS, the one for which a power allocation leading to minimum coherence is available, must be selected.

Firstly, in Algorithm 1, the parameters K and λ are determined. Secondly, the first sub-carrier is selected as the location of the first pilot. Afterwards, to extract the desired patterns, numbers between 2 and N are sequentially assigned to the next pilot locations. Each assignment is followed by calculating the difference multiset, $\mathcal{D}_{temp}$, of the current set of pilot locations. Then, the selected set of pilot locations is checked to see whether it is subset of some set forming CDS/ADS. This is done by checking two conditions:

$$\text{C.1}: \quad \forall a_d \in \mathcal{D}_{temp},\ a_d < \lfloor\lambda\rfloor + 2$$
$$\text{C.2}: \quad \#\{a_d \in \mathcal{D}_{temp}, a_d = \lfloor\lambda\rfloor+1\} < N-K$$

If both C.1 and C.2 are satisfied, the trend is continued for the next pilot locations. Otherwise, the assignment is not feasible and the remaining sub-carriers are checked until a proper set is found for the current pilot location. After the successful assignment of the first n pilot locations, if no proper sub-carrier is available for the $(n+1)$th pilot location, the algorithm backtracks and continues to find the next proper sub-carrier for the nth pilot location and so on.

In Appendix B, it is shown that a pattern of N_P pilots forms a CDS/ADS, if and only if, it satisfies conditions C.1 and C.2. Hence, pilot patterns that do not lead to a CDS/ADS will be excluded as soon as they do not satisfy at least one of the aforementioned conditions.

Algorithm 1: Finding pilot pattern & pilot power

Input: (N, N_P, max_{iter})
Output: $(\mathcal{P}, \mathbf{t})_{opt}$

calculate: $\lambda = \frac{N_P(N_P-1)}{N-1}$, $K = (N-1)(\lfloor\lambda\rfloor+1) - N_P(N_P-1)$
initialize: **flag** $\leftarrow$ **0**, $\mathcal{D} \leftarrow$ **0**, $\mathcal{P} \leftarrow \{1\}$, $i \leftarrow 1$, $g_{min} \leftarrow 1$
for $l = 1$ to max_{iter}
  $i \leftarrow i+1$
  for $j = p_{i-1}+1$ **to** $N - N_P + i$
    calculate difference multiset $\mathcal{D}_{temp}$ of $\mathcal{P} \cup \{j\}$
    if $\mathcal{D}_{temp}$ satisfies conditions C.1 and C.2
      $\mathcal{P} \leftarrow \mathcal{P} \cup \{j\}$, $\mathcal{D} \leftarrow \mathcal{D}_{temp}$
      if $|\mathcal{P}| = N_P$
        $\hat{\mathbf{t}} \leftarrow \texttt{fminsearch}(g(\mathcal{P}, \mathbf{t}), [1, 1, \cdots, 1]^T)$
        if $g(\mathcal{P}, \hat{\mathbf{t}}) < g_{min}$
          $g_{min} \leftarrow g(\mathcal{P}, \hat{\mathbf{t}})$, $(\mathcal{P}, \mathbf{t})_{opt} \leftarrow (\mathcal{P}, \hat{\mathbf{t}})$
        end if
        $\mathcal{P} \leftarrow \{p_1, p_2, \cdots, p_{i-1}\}$, **continue**
      end if
      $flag_i \leftarrow 1$, **break**
    end if
  end for (j)
  if $flag_i = 0$
    $flag_{i-1} \leftarrow 0$, $\mathcal{P} \leftarrow \{p_1, p_2, \cdots, p_{i-2}\}$, $i \leftarrow i-2$
  end if
end for (l)

Once a pattern forming CDS/ADS is found, the process of power allocation to the corresponding pilots will be initiated, i.e. the value of $\hat{\mathbf{t}}$ minimizing the coherence will be calculated numerically. Among several optimization methods, we selected *Nelder-Mead*, a non-linear optimization technique, which is well implemented in the $\texttt{fminsearch}(f(\mathbf{t}), \mathbf{t}_0)$ function of MATLAB. In this technique, which is suitable for non-smooth functions, the fast simplex search method is used to find the local minimum of the function $f(\mathbf{t})$ starting from the initial point $\mathbf{t}_0$[13]. Finally, if the current pair $(\mathcal{P}, \hat{\mathbf{t}})$ leads to a coherence less than the minimum coherence achieved so far, its value will be updated.

It was observed that the algorithm converged after a few number of iterations compared with the number of all possible pilot patterns (Fig. 4). Nevertheless, the algorithm is provided with an input max_{iter} that limits its running time. Note that the algorithm is not applicable for the pairs of (N, N_P) where neither CDS nor ADS exists. Fortunately, an exhaustive search over a large space showed that this happens in less than 5% of the cases. Therefore, the problem can be easily solved with a slight modification of N_P.

IV. Simulation Results

In Fig. 1, the coherence obtained by different schemes with respect to the number of pilots for an FFT size of $N = 256$ is depicted. It demonstrates that the proposed scheme's efficiency improved as the number of pilots increased. Also, pilot patterns obtained using the different schemes for $N_P = 13$ are included. It should be noted that for the proposed method, the height of the bars represents the power of different pilots normalized

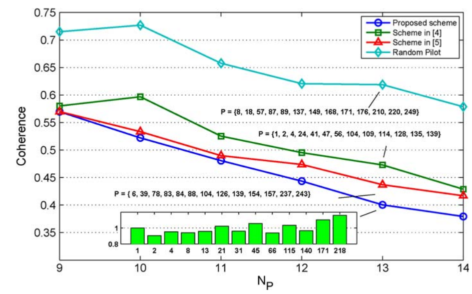

Fig. 1. Comparison of coherence obtained using the different schemes for $N = 256$.

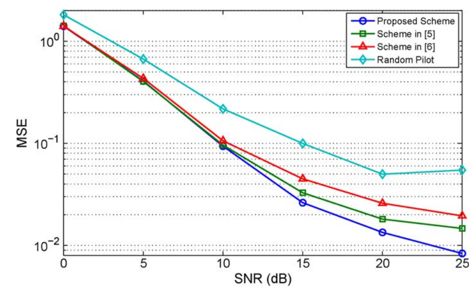

Fig. 2. Performance of the different schemes in channel estimation for $(N, N_P) = (256, 14)$.

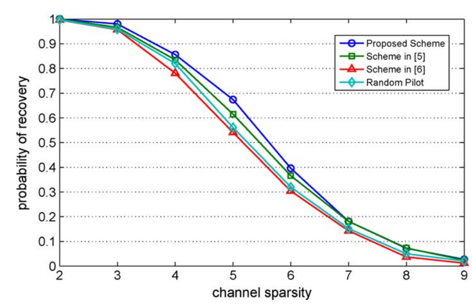

Fig. 3. Reconstruction probability for $(N, N_P) = (256, 14)$.

to unity. In the remaining simulations, the proposed scheme is compared with the scheme in [5] which outperformed [4]; the scheme in [6], which is based on a criterion similar to coherence; and the random pilot pattern.

Figs. 2 and 3 depict a comparison of the performance of the different schemes in practice. In these simulations 5000 realizations of sparse multipath channels with $L = 60$ for $N = 256$ are used. Position of non-zero channel taps are generated randomly and the attenuation is according to the independent and identically distributed (i.i.d.) $\mathcal{CN}(0, 1)$. Orthogonal matching pursuit (OMP) is the sparse recovery technique used throughout the simulations. In Fig. 2, sparse multipath channels with 3 non-zero taps are used to compare mean squared error (MSE) of the channel estimation for the different schemes. As it is shown, a more accurate estimation of the channel is achieved, using the proposed scheme. Fig. 3 compares the recovery percentage with respect to the channel sparsity (number of channel non-zero taps) for SNR = 22 dB. It indicates how the different schemes are robust against the sparsity of the channel. Also, in Fig. 4, the convergence speed of the proposed scheme is compared with its counterpart [5]. Note that in the other schemes the solution is obtained once and no updating is done, so they are not included in this simulation. Since the search space of the

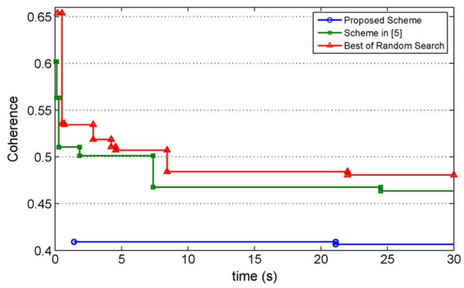

Fig. 4. Comparison of convergence speed for $(N, N_P) = (256, 13)$.

TABLE II
RUN-TIME OF THE DIFFERENT SCHEMES FOR $(N, N_P) = (256, 14)$

Scheme	Proposed Scheme	[4]	[5]	[6]
Run-time (s)	4.2129	0.3875	7.0356	2.5826

proposed algorithm is constrained to near-optimal patterns, it performs well from the beginning.

Table II lists the computer run-time of the different schemes for $(N, N_P) = (256, 14)$. Fig. 4 and Table II show that the proposed scheme outperforms that of [5] not only in efficiency, but also in convergence speed. However, the iterative nature of the scheme made it slower than that of [4] and [6]. It should be noted that this procedure is going to be done offline and once for all, so it is the efficiency, not the speed, that plays the key role. The simulation environment is based on a computer with 2 × 2.40 GHz Core i5 CPU and 4 GB memory.

V. CONCLUSION

In this paper, we investigated the deterministic design of pilot power and pattern in OFDM systems based on minimizing the coherence of the sensing matrix. In the proposed scheme, to obtain a solution, the pattern and power of pilots were jointly optimized. First, a search was performed to gather pilot patterns forming CDS/ADS. Then, power was optimally allocated to the different pilots in those patterns. Finally, the pair of pilot pattern and power leading to the minimum coherence was selected from the available pairs. Various simulation results validated the improvement of sparse estimation of the channel using the proposed scheme.

APPENDIX A

Let's prove it by contradiction. Assume that the multiset $\mathcal{D} = \{a_d\}_{d=1}^{N-1}$ has the minimum variance and there exist some a_{d_1} and a_{d_2} such that $a_{d_1} - a_{d_2} > 1$. Now, one can consider $\mathcal{D}^* = \{a_d^*\}$ equal to $\mathcal{D}$, but with $a_{d_1}^* = a_{d_1} - 1$ and $a_{d_2}^* = a_{d_2} + 1$,

$$\begin{aligned}&\mathrm{var}(\mathcal{D}) - \mathrm{var}(\mathcal{D}^*)\\ &\quad= \frac{1}{N-1}\left[(a_{d_1}-\lambda)^2+(a_{d_2}-\lambda)^2-\left(a_{d_1}^*-\lambda\right)^2-\left(a_{d_2}^*-\lambda\right)^2\right]\\ &\quad= \frac{2}{N-1}(a_{d_1}-a_{d_2}-1) > 0\end{aligned}$$

This contradicts the hypothesis; therefore, a_d's can differ at most by unity. By taking $\sum_{d=1}^{N-1} a_d = N_P(N_P-1)$ into account, they can only take on the values $\lfloor\lambda\rfloor$ and $\lfloor\lambda\rfloor + 1$. Hence, $\mathcal{D}$ is the difference multiset of either $(N, N_P, \lfloor\lambda\rfloor, K)$-ADS or (N, N_P, λ)-CDS for $\lambda \in \mathbb{Z}$, where K is the number of elements of $\mathcal{D}$ taking on the value $\lfloor\lambda\rfloor$ and is determined as follows:

$$\begin{aligned}&K\lfloor\lambda\rfloor + (N-1-K)(\lfloor\lambda\rfloor+1) = N_P(N_P-1) \Rightarrow\\ &\qquad\qquad K = (N-1)(\lfloor\lambda\rfloor+1) - N_P(N_P-1)\end{aligned}$$

APPENDIX B

Obviously, the converse is true, i.e. a set forming CDS/ADS satisfies conditions C.1 and C.2. Now, we show that the difference multiset $\mathcal{D}$ of some set $\mathcal{P}$ with N_P elements, cannot both satisfy conditions C.1 and C.2 and have elements less than $\lfloor\lambda\rfloor$. According to condition C.1, it will not have any element greater than $\lfloor\lambda\rfloor + 1$. Assume that it has $V \le N-K-1$ elements equal to $\lfloor\lambda\rfloor + 1$ (according to condition C.2), $M > 0$ elements with value less than $\lfloor\lambda\rfloor$, and $N-1-V-M$ elements equal to $\lfloor\lambda\rfloor$. Therefore:

$$\begin{aligned}\sum_{d=1}^{N-1} a_d &= V(\lfloor\lambda\rfloor+1) + \sum_{i=1}^{M} a_{d_i} + (N-1-M-V)\lfloor\lambda\rfloor\\ &= V + (N-1)(\lfloor\lambda\rfloor) + \sum_{i=1}^{M} a_{d_i} - M\lfloor\lambda\rfloor\\ &< V + (N-1)(\lfloor\lambda\rfloor) \le N_P(N_P-1).\end{aligned}$$

which is a contradiction, since for any set of N_P elements $\sum_{d=1}^{N-1} a_d = N_P(N_P-1)$. Hence, if conditions C.1 and C.2 are satisfied then $M = 0$ and a_d's will only take values $\lfloor\lambda\rfloor$ and $\lfloor\lambda\rfloor + 1$. Thus, $\mathcal{P}$ must realize an ADS or CDS.

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