Errorless Codes For CDMA Systems with Near-Far Effect

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Abstract—In this paper we propose a new model for the near-far effect in a CDMA system. We derive upper and lower bounds for the maximum near-far effect for errorless transmission. Using these bounds, we propose some near-far resistant codes. Also a very low complexity ML decoder for a subclass of the proposed codes is suggested.

I. INTRODUCTION

In a DS-CDMA system, every user is assigned a signature vector for transmitting data through a common channel (in time and frequency). In the channel all the transmitted vectors are added up and the resultant vector is observed at the receiver. In a synchronous CDMA system in the presence of noise, the channel model is

\[ Y = CX + N \]  \hspace{1cm} (1)

where \( C \) is the signature matrix, \( X \) is the user input vector with \{±1\}-entries and \( N \) is the additive noise vector.

Without perfect power control, the assumption of receiving +1 or −1 from each user is no more valid. The channel model for such near-far effect is

\[ Y = CAX + N \]  \hspace{1cm} (2)

where \( A = \text{diag}(a_1,\ldots,a_n) \) is the matrix of the received amplitude corresponding to each user.

Most of the works on compensating the near-far effect in CDMA are about introducing new methods for decoding such as MMSE in combination with Successive Interference Cancellation (SIC) [1] and blind adaptive interference suppression [2]. In [3], the method of Isolation Bit Insertion (IBI) is introduced for combating the near-far effect. In [4], the authors have found upper and lower bounds for near-far effect in a CDMA system. We derive upper and lower bounds for the near-far effect for errorless transmission. Using these bounds, we propose some near-far resistant codes. Also a very low complexity ML decoder for a subclass of these codes is suggested.

II. THE CHANNEL MODEL

In a CDMA system with no near-far effect, the diagonal matrix \( A \) discussed in the previous section is the identity matrix. For modeling the near-far effect in a CDMA channel, we assume that the received power of each user is bounded. Thus, every entry of the diagonal of the matrix \( A \) belongs to the interval \([1−η,1+η]\) for an \( η < 1 \) (because of the symmetry among the users, \( η \) can be assumed to be independent from the user index). We define the Near-Far Factor (NFF) of a CDMA system by

\[ \text{NFF} = 20\log_{10}\frac{\min p_i}{\max p_i} \]

where \( \max p_i \) and \( \min p_i \) are the maximum and the minimum received power for each user. Thus, in our model, we have

\[ \text{NFF} = 20\log_{10}\frac{1−η}{1+η} \]

A CDMA system with near-far effect with a noisy channel, can be modeled as

\[ Y = C(X + Z) + N \]  \hspace{1cm} (3)

where \( C \) is an \( m \times n \) signature matrix with normalized columns, \( X \) is an \( n \times 1 \) user data vector. \( Z \) is a random vector.
with entries belonging to the interval \([-\eta, \eta]\), and \(N\) is the noise vector. It is equivalent to (2), if we set \(a_i\)'s random variables with values in \([-\eta, \eta]\). In this paper, we concentrate on the noiseless case so the model reduces to

\[
Y = C(X + Z)
\]  

(4)

A natural problem arises about the maximum near-far effect in a CDMA system that does not affect unique decodability. Obviously, it depends on the signature matrix \(C\). According to [5], errorless communication is possible if the matrix \(C\) introduces a one-to-one mapping on \([\pm 1]^n\). In the presence of near-far effect, errorless communication is possible if the \(2^n\) \(m\)-dimensional shapes \(C \cdot (X + [-\eta, \eta]^n)\), \(X \in \{\pm 1\}^n\) are mutually disjoint (Fig. 1).

![Fig. 1. The 3-dimensional data vectors with near-far effect and the corresponding 2-dimensional received vectors.](image)

**Definition** Define

\[
\eta_{\text{sup}}(C) = \sup \{\eta | C \cdot (X + [-\eta, \eta]^n) \text{ are disjoint for all } X \in \{\pm 1\}^n\}.
\]

consequently,

\[
\text{NFF}_{\text{inf}} = 20 \log \frac{1 - \eta_{\text{sup}}}{1 + \eta_{\text{sup}}}
\]

In the following section, we find upper and lower bounds for \(\eta_{\text{sup}}(C)\).

### III. UPPER AND LOWER BOUNDS THEOREMS

In this section, we will find an upper bound for \(\eta\) under the assumption of errorless transmission.

**Theorem 1** For every \(m\) and \(n\) we have,

\[
\eta_{\text{sup}}(C_{m \times n}) \leq \frac{1}{2^m - 1}  
\]

(5)

**Proof:** Assume that the image of the cube \([-1, +1]^n\) by the linear transformation \(CX\) is \(F\). Suppose that the \(m\)-dimensional volume of \(F\) is \(v\). Since the channel is errorless, the \(n\)-dimensional cube with the side length \(2\eta\) around the points \((\pm 1)^n\) must be mapped to non-overlapping shapes. Because all these shapes are placed in the image of the cube with side length \(2 + 2\eta\), we have

\[
\text{Volume } (C \cdot [-1 - \eta, +1 + \eta]^n) \geq 2^n \text{ Volume } (C \cdot [-\eta, \eta]).
\]

Thus,

\[ (1 + \eta)^m v \geq 2^n \eta^m v. \]

In the following, we derive a lower bound for \(\eta\) for the signature matrix \(C\).

**Theorem 2** For any norm \(\| \|\) on \(\mathbb{R}^m\),

\[
\eta_{\text{sup}}(C) \geq \frac{\min_{x \in S} \| CX \|}{\max_{x \in [-1,1]^n} \| CX \|}  
\]

(6)

where \(S\) is the set of all non-zero elements of \([0, \pm 1]^n\).

**Proof:** If there exists \(X_1, X_2 \in \{\pm 1\}^n\) such that \(C \cdot (X_1 + [-\eta, \eta]^n) \cap C \cdot (X_2 + [-\eta, \eta]^n) \neq \emptyset\), there exists \(Z_1, Z_2 \in [-\eta, \eta]^n\) such that \(C(X_1 + Z_1) = C(X_2 + Z_2)\). Thus,

\[
C \left( \frac{X_1 - X_2}{2} \right) = C \left( \frac{Z_1 - Z_2}{2} \right). 
\]

Hence, the equation is formed as \(CX^* = CZ^*\) where \(X^* \in S\) and \(Z^* \in [-\eta, \eta]^n\). For the proof, we show that if \(\eta = \min_{x \in [-1,1]^n} \| CX \|\) - \(\varepsilon\), for any \(\varepsilon > 0\), there are no such \(X^*\) and \(Z^*\). Obviously,

\[
\max_{x \in [-1,1]^n} \| CX \| = \max_{x \in [-1,1]^n} \| CX \|
\]

and because the set \(C \cdot [-1,1]^n\) is the convex hull of \(C \cdot \{\pm 1\}^n\)

\[
\max_{x \in [-1,1]^n} \| CX \| = \max_{x \in [-1,1]^n} \| CX \|
\]

Thus,

\[
\max_{x \in [-1,1]^n} \| CX \| = \left( \frac{\min_{x \in S} \| CX \|}{\max_{x \in [-1,1]^n} \| CX \|} - \varepsilon \right) \max_{x \in [-1,1]^n} \| CX \| < \min_{x \in S} \| CX \|
\]

which means there are not \(X^* \in S\) and \(Z^* \in [-\eta, \eta]^n\) that \(CX^* = CZ^*\). Q.E.D.

This theorem proves that errorless transmission is possible for any \(\eta\) such that

\[
\eta < \sup_{\| \|} \left( \frac{\min_{x \in S} \| CX \|}{\max_{x \in [-1,1]^n} \| CX \|} \right)
\]

**Example 1** For the matrix

\[
C_{2 \times 4} = \begin{bmatrix}
1 & 0 & 0.5\sqrt{3} & -0.5 \\
0 & 1 & 0.5 & 0.5\sqrt{3}
\end{bmatrix}
\]

the upper bound of Theorem 1 says that \(\eta_{\text{sup}}(C_{2 \times 4}) \leq 0.33\). We simulated the lower bound of Theorem 2 for 100 random norms on \(\mathbb{R}^2\) and found that \(\eta_{\text{sup}}(C_{2 \times 4}) \geq 0.18\). Actually for generating a random norm, we chose a random matrix \(A\) and different \(p > 1\), define \(\|X\|_p = \|AX\|_p\) where \(\| \|_p\) denotes the \(p\)-norm. Computer simulation show that \(\eta_{\text{sup}}(C_{2 \times 4}) \approx 0.21\) (NFF_{inf} \(\approx -3.75\)dB), which is very close to both upper and lower bounds.
Proof: Assume $\eta > \eta_{\text{sup}}(C)$. According to the proof of Theorem 2, there exist $X^* \in S$ and $Z^* \in [-\eta, \eta]^n$ such that $C X^* = C Z^*$. Let $\tilde{X} = [X^* \, \ldots \, X^*]^T \in \{0, \pm 1\}^{kn}$ and $\tilde{Z} = [Z^* \, \ldots \, Z^*]^T \in [-\eta, \eta]^{kn}$, we have,

$$
(P^{-1} \otimes I)(P \otimes C)\tilde{X} = (I \otimes C)\tilde{X} = (I \otimes C)\tilde{Z} = (P^{-1} \otimes I)(P \otimes C)\tilde{Z}.
$$

Thus, $(P \otimes C)\tilde{X} = (P \otimes C)\tilde{Z}$. Consequently, $\eta_{\text{sup}}(P \otimes C) \leq \eta_{\text{sup}}(P \otimes C)$. Now, consider $\eta > \eta_{\text{sup}}(P \otimes C)$ there exist non-zero $\tilde{X} \in \{0, \pm 1\}^{kn}$ and $\tilde{Z} \in [-\eta, \eta]^{kn}$ such that $(P \otimes C)\tilde{X} = (P \otimes C)\tilde{Z}$. Thus, $(I \otimes C)\tilde{X} = (P^{-1} \otimes I)(P \otimes C)\tilde{X} = (P^{-1} \otimes I)(P \otimes C)\tilde{Z} = (I \otimes C)\tilde{Z}$. Let $X^*$ and $Z^*$ be the first $n$ entries of $\tilde{X}$ and $\tilde{Z}$, respectively. We have $C X^* = C Z^*$, which means $\eta_{\text{sup}}(P \otimes C) \geq \eta_{\text{sup}}(C)$.

Note: Using Theorems 3 and 2, for every invertible matrix $P$, we derive another lower bound

$$
\eta_{\text{sup}}(C) \geq \min_{X \in \{0, \pm 1\}^{kn}} \| (P \otimes C)X \|_{\infty},
$$

In the next section, we propose a very low complexity ML decoder for a subclass of the proposed codes.

V. MAXIMUM LIKELIHOOD (ML) DECODER FOR A CLASS OF THE PROPOSED NEAR-FAR RESISTANT CODES

In [5] a very low complexity method for decoding COW codes is proposed. Here we try to extend those ideas to decode near-far resistant matrices. In this section we present a suboptimum decoding method that has a very low computational complexity. Also, we study the conditions under which this decoder is ML.

As (3), we model a synchronous CDMA system with near-far effect in an AWGN channel as

$$
Y = CX + Z + N
$$

Here $N$ is a Gaussian vector with zero mean and auto-covariance matrix $\sigma^2 I$ ($I$ is the identity matrix).

Since each entry of $Z$ is a random number belonging to $[-\eta, \eta]$, for the worst case, it has uniform distribution in the interval $[-\eta, \eta]$ (a compact support random variable has maximum entropy when its distribution is uniform [7]). Also, we assume that the time samples of $Z$ are independent. This assumption is much worse than what occurs in practice because in practical situations the power of each user is constant at least for a short period of time and thus the time samples are correlated.

For overloaded systems, the generalized central limit theorem suggests that we can approximate $W = CZ + N$ with a Gaussian vector with zero mean and auto-covariance matrix $\frac{\eta^2}{3} CC^T + \sigma^2 I$. This approximation becomes better as the overloading factor grows [8]. Thus, hereafter, we model the channel as

$$
Y = CX + W
$$

where $W$ is a zero mean Gaussian random vector with covariance matrix $\frac{\eta^2}{3} CC^T + \sigma^2 I$.

Similar to [5-6], we prove a lemma that significantly decreases the complexity of the decoding problem of Theorem 3.

Lemma 1 Suppose $P$ is an invertible matrix and $D_{km \times kn} = P_{k \times k} \otimes C_{m \times n}$. The decoding problem of a system with the signature matrix $D$ can be decoupled to $k$ decoding of a system with the signature matrix $C$.

Proof: Suppose $Y = (P \otimes C)X + W$. Multiplying both sides by $P^{-1} \otimes I_m$, we have

$$
(P^{-1} \otimes I_m)Y = (I_k \otimes C)X + (P^{-1} \otimes I_m)W.
$$

Since $I_k \otimes C$ is a block diagonal matrix, the decoding problem of $P^{-1} \otimes I_m$ is in fact $k$ disjoint decoding problems. Now, if the matrix $P$ is unitary, then $P^{-1} \otimes I_m$ is also a unitary matrix and thus the distribution of the vector $(P^{-1} \otimes I_m)W$ is the same as the distribution of $W$. Consequently, the ML decoding of these $k$ small size problems is equivalent to the ML decoding of the big size matrices.

The following lemma reduces the decoder complexity even more.

Lemma 2 Suppose $C_{m \times n}$ is full rank. The decoding problem for a system with the signature matrix $C$ can be performed by $2^{n-m}$ Euclidean distance calculations instead of $2^n$.

Proof: Since $C$ is full rank, we can assume that $C = [A | B]$ where $A$ is an $m \times m$ invertible matrix. Thus, $Y = AX_1 + BX_2 + W$ where $X_1$ and $X_2$ are $m \times 1$ and $(n-m) \times 1$ vectors, respectively. Since $A^{-1}Y = X_1 + A^{-1}BX_2 + A^{-1}W$
and sign(V) (that is obtained by substituting the positive entries of a vector V by +1 and the negatives by −1) is the nearest {±1}-vector to V, equations

\[ \hat{X}_2 = \arg\min_z \| (A^{-1}Y - A^{-1}BZ) - \text{sign}(A^{-1}Y - A^{-1}BZ) \|_2 \]

\[ \hat{X}_1 = \text{sign}(A^{-1}Y - A^{-1}B\hat{X}_2) \]

minimize \( \| A^{-1}Y - A^{-1}B\hat{X}_2 - \hat{X}_1 \|_2 \) which is a suboptimum decoder. This method needs \( 2^{n-m} \) Euclidean norm calculations. It is worth mentioning that if A is a unitary matrix, then the above algorithm is optimum [5].

This lemma suggests a decoder with lower complexity; however, in general, this is a suboptimum decoder unless A is unitary and \( CC^T \) is a scaled identity matrix (this means that its rows must be orthogonal to each other, i.e., a WBE matrix [9]).

**Example 3** From Theorem 3 and Example 1, \( D_{64 \times 128} \) is a matrix with \( \eta_{\sup}(D_{64 \times 128}) \approx 0.21 \), i.e., \( \text{NFF}_{\text{inf}}(D_{64 \times 128}) \approx -3.7 \text{ dB} \). Since \( I_{32} \) is a unitary matrix, the first two columns of \( C \) is also a unitary matrix and the rows of \( C \) are orthogonal to each other, lemmas 1 and 2 result in ML decoder of a system with signature matrix \( D_{64 \times 128} \). The proposed method of decoding needs \( 32 \times 2^2 \) Euclidean norm calculations instead of \( 2^{128} \) such calculations in direct implementation of the ML decoder. The performance of this code in an AWGN channel is simulated in the next section.

**Example 4** Similar to Example 3 for \( D_{64 \times 104} = H_{8 \times 8} \otimes C_{8 \times 13} \), where \( H_{8 \times 8} \) is an 8 \( \times \) 8 Hadamard matrix and \( C_{8 \times 13} \) is the matrix in Example 2, \( \eta_{\sup}(D_{64 \times 104}) \approx 0.23 \), i.e., \( \text{NFF}_{\text{inf}}(D_{64 \times 104}) \approx -4 \text{ dB} \). The advantage of this matrix is that its entries are \( \pm 1 \) (it is in fact a COW matrix). Since \( \frac{1}{\sqrt{8}}H_{8 \times 8} \) is a unitary matrix, according to Lemma 1, the ML decoder of \( D_{64 \times 104} \) can be implemented by 8 ML decoder of \( C_{8 \times 13} \). This means significant reduction in the complexity of the ML decoder; \( 8 \times 2^3 \) Euclidean norm calculations instead of \( 2^{104} \) such calculations. However, using Lemma 2, we obtain a suboptimum decoder with \( 8 \times 2^5 \) Euclidean norm calculations. This decoder is not ML because the rows of \( C_{8 \times 13} \) are not orthogonal but it is somewhat suboptimum.

** VI. SIMULATION RESULTS**

We have simulated two overloaded binary (64,128) and (64,104) CDMA systems. The code matrices used for these simulations are \( D_{64 \times 104} \) and \( D_{64 \times 128} \) which are introduced in Examples 3 and 4. The advantage of the system with 104 users is that its signature matrix is binary antipodal which is practically favorable. In our simulations we have assumed the near-far effect for each user is a white random process, i.e., there is no correlation among the time samples. Obviously, this scenario is much worse than what occurs in practical situations. The advantage of this decoding method is that we have assumed that the receiver does not have any knowledge about the received user powers. Notice that in such highly over-loaded systems, conventional methods for estimating the received powers do not work. However, the performance of the system in the presence of noise and near-far effect is very impressive. Simulations are performed for various values of NFF. As it can be predicted, Figs. 2 and 3 show that for NFF more than NFF_{inf} the Bit-Error-Rate (BER) tends to zero as \( E_b/N_0 \) grows and for NFF less than NFF_{inf}, BER saturates and errorless transmission is not possible. The simulation results show the robustness of the proposed codes against additive noise and near-far effects.

Both systems employ the proposed decoder with low complexity. The decoder is ML for the system with 128 users and is sub-optimum for the system with 104 users.

![Fig. 1. BER versus E_b/N_0 for a binary CDMA system with 64 chips and 104 users (binary signatures/binary inputs).](image)

The following simulation is for a non-binary \( D_{64 \times 128} \) matrix. Because this matrix constructed by Kronecker product of identity matrix with the \( 2 \times 4 \) WBE matrix, the decoder of Lemma 2 in the previous section is an ML decoder.

![Fig. 2. BER versus E_b/N_0 for a non-binary CDMA system with 64 chips and 104 users (binary signatures/binary inputs).](image)
VII. CONCLUSION AND FUTURE WORKS

In this paper, we have considered highly overloaded CDMA systems that are robust against near-far effects. The systems are errorless in noiseless channels provided that the near-far effect is lower than a desired value. Also, we have proposed an optimum and suboptimum decoder for a class of the proposed codes. Upper and lower bounds have been derived for the maximum value of near-far effect that does not affect the errorless property of the system. The assumptions in this paper are much worse than what actually occurs in a real situation. We assume that the received user powers are white random processes (there is no correlation among the time samples). However, under such a bad condition, the proposed systems show acceptable performances as demonstrated in Figs. 2 and 3.

For future work, we plan to consider the correlation among the time samples of the near-far effects in practical situations. This assumption will improve the performance of the system. Future work should focus on discovering codes that can tolerate more near-far effects; also the derivation of the sum channel capacity of a CDMA system [10] with near-far effects is very useful.

REFERENCES


