Sampling and Distortion Tradeoffs for Bandlimited Periodic Signals

Elaheh Mohammadi and Farokh Marvasti

***Abstract*— In this paper, the optimal sampling strategies (uniform or nonuniform) and distortion tradeoffs for Gaussian bandlimited periodic signals with additive white Gaussian noise are studied. Our emphasis is on characterizing the optimal sampling locations as well as the optimal presampling filter to minimize the reconstruction distortion. We first show that to achieve the optimal distortion, no presampling filter is necessary for any arbitrary sampling rate. Then, we provide a complete characterization of optimal distortion for low and high sampling rates (with respect to the signal bandwidth). We also provide bounds on the reconstruction distortion for rates in the intermediate region. It is shown that nonuniform sampling outperforms uniform sampling for low sampling rates. In addition, the optimal nonuniform sampling set is robust with respect to missing sampling values. On the other hand, for the sampling rates above the Nyquist rate, the uniform sampling strategy is optimal.**

***Index Terms*— Sampling locations, nonuniform sampling, distortion.**

I. Introduction

SHANNON'S rate distortion theory finds the optimal compression rate for a given distortion for an i.i.d. discrete signal. For continuous signals, Shannon assumes that the signal is sampled (noiselessly) at the Nyquist rate. Thus, in Shannon's protocol, distortion occurs at the quantization phase; no information about the signal is discarded in the sampling phase. But since only the end-to-end distortion is of importance, one can accept some distortion at the sampling phase, by sampling the signal below the Nyquist rate, or by assuming sampling noise. In other words, we can discard information about the signal in both the sampling and quantization phases. Sampling at a sub-Nyquist rate has the benefit of saving in the sampling rate, which can improve power and computational efficiency of the system [1], [2]. Furthermore, choosing the sampling locations (*nonuniform sampling*), one can further reduce the sampling rate.

Nonuniform sampling works by measuring signal values at a set of arbitrary locations in time. This can occur when we are either unable to uniformly sample the signal due to some physical constraints, or when we lose some of the samples after uniform sampling. Furthermore, nonuniform sampling is helpful in dealing with aliasing [3]. Similar to uniform sampling, it has been shown that for reliable recovery, the average sampling rate must be at least twice the bandwidth of the signal [4]. See [5] and [6] for an overview of nonuniform sampling.

Consider the problem of finding the best locations for sampling a continuous signal to minimize the reconstruction distortion. To find the optimal sampling points, one should utilize any available prior information about the structure of the signal. Furthermore, sampling noise should be taken into account. Due to its importance, nonuniform sampling and its stability analysis has been the subject of numerous works, in particular for deterministic signals (for instance, see [7]). However, there has been relatively less work for a fully Bayesian model of the signal and sampling noise, where one is interested in the statistical *average* distortion of signal reconstruction over a class of signals. Some previous works along this line address the problem of finding the best locations of nonuniform sampling for minimizing reconstruction distortion. These works can be categorized according to their signal and sampling models. These works generally do not consider sampling of *periodic bandlimited signals*, which is the topic of this paper. A summary of these works and their comparison with our work is provided later in the introduction.

A. System Model

Unlike the previous works that consider aperiodic Gaussian stationary sources with a given autocorrelation function, or aperiodic Markov sources, herein we consider a *stochastic continuous periodic signal* with period T. Any such signal is characterized by its values in one period $[0, T]$. Conversely, from a practical perspective, if we are interested in a signal that is defined *only* on a finite interval $[0, T]$, we can periodically extend it and view it as a periodic signal with period T. Similarly, a finite set of M nonuniform samples of the signal in $[0, T]$ can be periodically extended to generate a set of periodic nonuniform samples.

In this paper, we further assume the signal to be bandlimited, with at most $2(N_2 - N_1 + 1)$ non-zero Fourier series coefficients from the frequency $N_1\omega_0$ to $N_2\omega_0$, where $\omega_0 = 2\pi/T$ is the fundamental frequency and T is the signal period, *i.e.,*

$$S(t) = \sum_{\ell=N_1}^{N_2} [A_\ell \cos(\ell\omega_0 t) + B_\ell \sin(\ell\omega_0 t)], \quad t \in [0, T]. \tag{1}$$

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The authors are with the Department of Electrical Engineering, Advanced Communications Research Institute, Sharif University of Technology, Tehran 11365-11155, Iran.

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There are $2N \triangleq 2(N_2 - N_1 + 1)$ free variables A_ℓ and B_ℓ for $\ell \in [N_1 : N_2]$. To obtain information about the signal, we can assume that all of the samples are taken at time instances t_i in the interval $[0, T]$. A noiseless sample at time $t_i \in [0, T]$, $S(t_i)$ is a linear equation of coefficients A_ℓ and B_ℓ for $\ell \in [N_1 : N_2]$. The $2N$ free variables A_ℓ and B_ℓ can be uniquely reconstructed from $M = 2N$ linearly independent equations. But we consider taking M *noisy* samples $S(t_i) + Z_i$ (at time instances t_i that we choose) with sampling noise Z_i. The sampling noise can model quantization noise of an A/D convertor, or the noise incurred by transmitting the samples to a fusion center over a communication channel. Perfect reconstruction from noisy samples is not in general feasible, and distortion is inevitable. To minimize the distortion given a maximum number of M samples, we optimize over sampling locations t_i as well as a pre-sampling filter that we consider in our model. See Fig. 1 for a description of our problem.

The Fourier series coefficients (A_ℓ and B_ℓ for $\ell \in [N_1 : N_2]$) and the sampling noise Z_i are all assumed to be normal random variables.[1] Therefore, both the input signal $S(t)$ and its reconstruction $\hat{S}(t)$ are random signals, and the reconstruction distortion

$$\text{Sampling Distortion} = \frac{1}{T}\int_0^T |S(t) - \hat{S}(t)|^2 dt,$$

is also a random variable. We wish to minimize the Sampling Distortion, which is a random variable. To minimize the values that a random variable may take, besides minimizing its mean, we also wish to quantify how far the distribution of the random variable is spread out around its mean. The literature in sampling theory only looks at minimizing the *expected value* of the Sampling Distortion. However, we are considering *variance* of the Sampling Distortion as a measure of the *degree of concentration* of the Sampling Distortion around its mean (as variance is a measure of how far random realizations are spread out from their mean). One of the contributions of this paper is to raise the possibility that minimizing the expected value of error may potentially increase its variance. In such cases, just minimizing MMSE may not be practically appealing if it leads to large error variance.[2]

Herein, we consider minimizing both the expected value and variance of the Sampling Distortion, by choosing the best sampling points and the pre-sampling filter. We denote the minimum of the expected value and variance of the Sampling Distortion by $\mathsf{D}_{\min}$ and $\mathsf{V}_{\min}$, respectively. A small $\mathsf{D}_{\min}$ guarantees a good *average* performance over all instances of the random signal, while a small $\mathsf{V}_{\min}$ guarantees that with high probability, we are close to the promised average distortion for a given random signal (see also Appendix A). In essence, our goal is to find the tradeoffs between the number of samples, M, the pre-sampling filter, the variance of noise, σ^2, and the optimal expected value and variance of distortion, $\mathsf{D}_{\min}$ and $\mathsf{V}_{\min}$.

[1] We can relax the Gaussian assumption and instead say that we are studying the minimization of the *linear MMSE* in this paper. For Gaussian distributions, MMSE and linear MMSE are identical. Also observe that the statistical bandpass assumption may be a good model for voiced speech, audio signals, and fading channel estimation for OFDM (having an almost periodic impulse response).

[2] It is possible to conceive other measures that quantify the degree of concentration of reconstruction error around its mean via for instance higher order statistics (or other functions of the pdf). However, the focus of our work is to pick the common measure of variance to make the point about the importance of just minimizing the expected value of error vs. also looking at its concentration around mean.

B. Overview of Main Results

To state the main results, let us make the following definitions based on the signal description given in equation (1):

- We call $(N_2 - N_1 + 1) f_0 = N f_0$ the *signal bandwidth* (of the bandlimited signal), where $f_0 = 1/T$.
- We call $2N_2 f_0$ the *Nyquist rate* (twice the maximum frequency of the signal).
- We call $M/T = M f_0$ the *sampling rate*. It is the number of total samples in a period $[0, T]$, divided by period length T. If we periodically extend the M samples (periodic nonuniform sampling), M/T will be the number of samples taken per unit time, hence called the sampling rate.

We provide tight results or bounds on the tradeoffs among various parameters such as distortion, sampling rate and sampling noise. We first show that to achieve the optimal average and variance of distortion, no pre-sampling filter is necessary for any arbitrary number of samples M. Next, when the sampling rate $M f_0$ is below the signal bandwidth $N f_0$, *i.e.,* $M \le N$, we find the optimal average and variance of distortion, denoted by $\mathsf{D}_{\min}$ and $\mathsf{V}_{\min}$, respectively. Interestingly, we show that the minima of both $\mathsf{D}_{\min}$ and $\mathsf{V}_{\min}$ are obtained at the same sampling locations. Minimum occurs if we sample at any arbitrary subset of size M of $\{0, T/N, 2T/N, 3T/N, \cdots, (N-1)T/N\}$. If $M < N$, this forms a *nonuniform sampling* set. It is worth to note that the sampling locations only depend on the bandwidth of the signal; they are optimal for all values of the noise variance σ^2 and N_1. Moreover, the sampling points are robust with respect to missing samples. Note that the optimal sampling points are any arbitrary M points from the set $\{0, T/N, 2T/N, 3T/N, \cdots, (N-1)T/N\}$. Thus, if we sample at these positions and we miss some samples, *i.e.,* getting $M' < M$ samples instead of M samples, the set of M' sampling points is still a subset of $\{0, T/N, 2T/N, 3T/N, \cdots, (N-1)T/N\}$, and hence optimal. Finally, complete characterization of distortion in terms of M and σ^2, allows us to answer the question of whether it is better to collect a few accurate samples from a signal, or collect many inaccurate samples from the same signal (a problem related to selecting an appropriate $\Sigma\Delta$ modulator).

For sampling rates $M f_0$ above the signal bandwidth $N f_0$, we provide lower and upper bounds that are shown to be tight in some cases. When $N f_0 < M f_0 \le 2N f_0$, we find the optimal average distortion when N divides $2N_1 - 1$, *i.e.,* $N | 2N_1 - 1$, using a non-uniform set of sampling locations. In addition, if the sampling rate $M f_0$ is above the Nyquist rate $2N_2 f_0$, *i.e.,* $M > 2N_2$, the uniform sampling is shown to be optimal under certain constraints. Whenever we find $\mathsf{D}_{\min}$ and $\mathsf{V}_{\min}$ explicitly, the minima are achieved simultaneously at the same optimal sampling points.

C. Proof Technique

Due to the assumption of Gaussian distribution on the signal coefficients A_ℓ and B_ℓ, minimizing the MMSE of the signal reduces to minimizing the linear MMSE (LMMSE). LMMSE can be expressed as a linear algebra optimization problem over matrices. However, this is a non-linear optimization problem because the coordinates of the matrices include sine and cosine functions of the sampling locations t_i (the variables we are optimizing over are sampling locations t_i). Sine and cosine functions are nonlinear, albeit structured, functions; their structure can be exploited to solve the optimization problem. A key tool that we repeatedly exploit is an inequality in *majorization theory* that relates trace of a function of a matrix to the diagonal entries of the matrix (see [8, Ch. 2] for an overview of majorization theory).

D. Related Works

As mentioned above, the signal model considered in previous works differs from our model. Nonetheless, we remark on similarities and differences among the conclusions of our work and the other papers. Sugiyama and Ogawa [9] consider the problem of learning of a function from its noisy samples. The function is assumed to be a low-pass *deterministic* signal (which differs from our assumption of a band-pass *random* signal with random Fourier coefficients). It is also assumed that the sampling rate is high so that when there is no sampling noise, there are enough equations for perfect reconstruction. This constraint narrows down the paper to the high sampling case. They ask for the optimal sampling locations and show that uniform sampling is optimal at high sampling rates. Even though the setup of [9] is different from ours but [9] also shows the optimality of uniform sampling for high rates.

A number of papers consider the reconstruction distortion of a *stationary process* from its noisy samples under the uniform sampling strategy. Reconstruction of a *lowpass* stationary process from its uniform samples was expressed in an early work [10] in terms of the samples of the auto-correlation function of the process. The same paper assumes no sampling noise, and shows that uniform sampling is sufficient above the Nyquist rate for perfect signal reconstruction. Assuming noisy uniform samples, optimal pre-sampling filter and the corresponding distortion has been found in [11]–[13] via different proof techniques.

Reconstruction distortion from noisy nonuniform samples has also been considered in the literature. Wu and Sun [14] consider a *stationary Gaussian signal model* with auto-correlation function $R(\tau) = \rho^{|\tau|}$. They adopt a non-adaptive sampling strategy and show that amongst all nonuniform sampling strategies, uniform sampling is optimal to minimize the average sampling distortion. This is in contrast with our results on the benefit of nonuniform sampling in the low sampling rate region. However, it relates to our result in the high sampling region where uniform sampling strategy is optimal. Thus, optimality of uniform sampling strategy depends on the class of stochastic signals one considers. Feizi *et al.* [1] assume a *first order Markov* signal. They consider an *adaptive sampling* method, wherein the location of the next sample is chosen based on the previous sampling locations and values. The paper employs dynamic programming and greedy techniques to solve this problem. This paper shows that adaptive nonuniform sampling outperforms uniform sampling. Finally, there are some papers such as [15]–[17] that consider specialized signal models (e.g. based on differential equations) to study sampling of a variable of interest in a given practical application. There are also less related works (such as [18]–[20]) that discuss the tradeoffs between sampling rate and reconstruction distortion in the context of quantization or compressed sensing.

E. Organization of the Paper

In Section II, the problem is formally defined. In Section III, the main results of the paper are presented; the proofs are relegated to Appendix B. In Section IV the problem formulation in matrix form as well as an overview of the proof techniques are given. In Section V, some remarks and extensions of our results are stated. Finally the proofs and technical details are given in the appendices.

II. Problem Definition

We consider a continuous bandlimited periodic signal defined as follows:

$$S(t) = \sum_{\ell=N_1}^{N_2} [A_\ell \cos(\ell\omega_0 t) + B_\ell \sin(\ell\omega_0 t)], \quad t \in [0, T] \tag{2}$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency. The summation is from $\ell = N_1$ to N_2, indicating that the signal is bandlimited. We assume that A_ℓ and B_ℓ for $N_1 \le \ell \le N_2$ are mutually independent Gaussian r.v.s, distributed according to $\mathcal{N}(0, \mathsf{p})$ for some $\mathsf{p} > 0$.[3] Thus, the signal power is $N\mathsf{p}$, where $N = N_2 - N_1 + 1$.

Our model is depicted in Figure 1. The signal $S(t)$, given in (2), is passed through a pre-sampling filter, $H(\omega)$, to produce $\tilde{S}(t)$. The signal $\tilde{S}(t)$ is sampled at time instances $t_1, t_2, \cdots, t_M$, where $t_i \in [0, T)$. These samples are then corrupted by $Z(t)$, an i.i.d. zero-mean Gaussian noise $\mathcal{N}(0, \sigma^2)$. Thus, our observations are $Y_i = \tilde{S}(t_i) + Z_i, i = 1, 2, \cdots, M$. An estimator uses the noisy samples $(Y_1, Y_2, \cdots, Y_M)$ to reconstruct the original signal, denoted by $\hat{S}(t)$. The incurred sampling distortion given by MMSE criterion is equal to

$$\text{Sampling Distortion} = \frac{1}{T}\int_0^T |\hat{S}(t) - S(t)|^2 dt.$$

This distortion is a random variable. Our goal is to compute the minima of the expected value and variance of this random variable, *i.e.,* to minimize

$$\mathsf{D} = \mathbb{E}\left\{\frac{1}{T}\int_0^T |\hat{S}(t) - S(t)|^2 dt\right\}, \tag{3}$$

$$\mathsf{V} = \mathsf{Var}\left\{\frac{1}{T}\int_0^T |\hat{S}(t) - S(t)|^2 dt\right\}. \tag{4}$$

[3] Observe that the term $A_\ell \cos(\ell\omega_0 t) + B_\ell \sin(\ell\omega_0 t)$ can be expressed as $C_\ell \cos(\ell\omega_0 t + \phi_\ell)$, where $C_\ell = \sqrt{A_\ell^2 + B_\ell^2}$. Since A_ℓ and B_ℓ are mutually independent Gaussian r.v.s with the same variance, C_ℓ has a Rayleigh distribution with zero mean, and ϕ_ℓ has a uniform distribution. Furthermore, C_ℓ and ϕ_ℓ are mutually independent. Hence, the signal is wide-sense stationary. Since the r.v.s are jointly Gaussian, it is also strict-sense stationary.

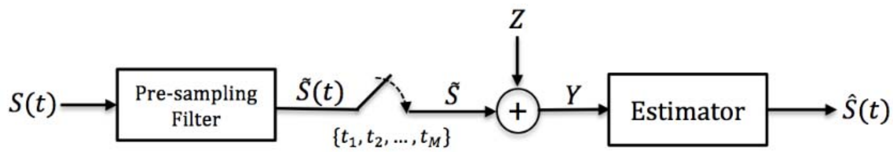


Fig. 1. Sampling-Distortion Model. The signal $S(t)$ is passed through the pre-sampling filter. The output $\tilde{S}(t)$ is sampled at times $t_1, t_2, \cdots, t_M$. A noise Z is added to the signal at the time of sampling. The noisy observations are then used to recover the signal.

We are free to choose $H(\omega)$ and sampling locations t_i, $i \in \{1, 2, \cdots, M\}$. Therefore, the optimal average and variance of distortion are defined as follows:

$$\mathsf{D}_{\min} = \min_{H(\cdot),t_1,t_2,t_3,\cdots,t_M} \mathbb{E}\left\{\frac{1}{T}\int_0^T |\hat{S}(t) - S(t)|^2 dt\right\}, \tag{5}$$

$$\mathsf{V}_{\min} = \min_{H(\cdot),t_1,t_2,t_3,\cdots,t_M} \mathsf{Var}\left\{\frac{1}{T}\int_0^T |\hat{S}(t) - S(t)|^2 dt\right\}, \tag{6}$$

where the minimization is taken over the sampling locations and the pre-sampling filter.

We assume that $H(\omega)$ is a real LTI filter such that

$$|H(\ell\omega_0)|^2 \leq 1 \quad for\ all \quad N_1 \leq \ell \leq N_2, \tag{7}$$

meaning that the frequency gain of the filter is at most one; in other words, we assume that the filter is passive and hence cannot increase the signal energy in each frequency. In particular, all-pass filters $|H(\omega)| = 1$ satisfy this assumption.[4]

III. Main Results

In this section, we first claim that no pre-sampling filter is necessary to achieve the optimal distortion, and then state our results on the optimal choices of the sampling locations. Pre-sampling filter can be potentially useful because it allows one to remove some of the frequency components at the beginning, and thereby enable the sampler to recover the remaining components with a lower estimation error. The total error will be equal to the variance of the filtered components, plus the estimation error of the unfiltered components. It turns out that such a strategy cannot outperform the naive strategy of not using any filter at all. Our first result states this fact:

Theorem 1: Consider the class of passive pre-sampling filters, i.e., *real filters $H(\omega)$ satisfying*

$$|H(\ell\omega_0)|^2 \leq 1 \quad for\ all \quad N_1 \leq \ell \leq N_2. \tag{8}$$

Then, the average and variance of the distortion are minimized by the identity filter $H(\omega) = 1$ for all ω (which is equivalent with not using any pre-sampling filter).

The above theorem implies that not using any pre-sampling filter is optimal. In particular, a pre-sampling filter that reduces the signal bandwidth to *half* the sampling rate (an anti-aliasing filter) is suboptimal. In Section V-A, we discuss this sub-optimality in more details. Besides this section, in the subsequent parts of the paper we mainly avoid discussion of the pre-sampling filter.

In the following, we state two general lower bounds on the average and variance of distortion and then, using these lower bounds, we provide the main results of the paper. Note that $SNR = N\mathsf{p}/\sigma^2$, and $N = N_2 - N_1 + 1$ which is proportional to the bandwidth of the signal (which is Nf_0). Remember that M is the number of noisy samples which is proportional to the sampling rate (defined as Mf_0). The proofs of the main results are given in Appendix B. We did not provide the formal proofs below the statement of the theorems because we first need to represent the problem in a matrix form, as given in Section IV-A and Appendix B-A.

A. Lower Bound 1 on Distortion

We begin by providing our first lower bounds on $\mathsf{D}_{\min}$ and $\mathsf{V}_{\min}$:

Lemma 1 (Lower Bound 1): For any noise variance $\sigma > 0$, the following lower bounds hold on the minima of the average and variance of the reconstruction distortion for any values of M, N and SNR:

$$\frac{\mathsf{D}_{\min}}{\mathsf{p}} \geq \frac{1}{2}\left(2N - M + \frac{M}{1+SNR}\right),$$
$$\frac{\mathsf{V}_{\min}}{\mathsf{p}^2} \geq 2N - M + \frac{M}{(1+SNR)^2}.$$

Furthermore, the above inequalities hold with equality for given values of M and N, if one can find a set of M distinct sampling times $\{t_1, t_2, \ldots, t_M\}$ such that for any $i \neq j$, time instances t_i and t_j satisfy at least one of the following two equations:

$$|t_i - t_j| = T\frac{m_1}{N}, \quad for\ some\ natural\ number\ m_1, \tag{9}$$

$$|t_i - t_j| = T\frac{2m_2 - 1}{2(N_1 + N_2)}, \quad for\ some\ natural\ number\ m_2. \tag{10}$$

Observe that when $M \leq N$, if we choose $t_i = iT/N$ for $i = 1, 2, \cdots, M$, then $|t_i - t_j| = |i - j|T/N$ is an integer multiple of T/N. Hence, the above condition can be satisfied for any $M \leq N$. On the other hand, this condition cannot hold if $M > 2(N_1 + N_2)$. This is because any set of distinct points $t_1, t_2, \ldots, t_M \in [0, T]$ will have two points whose difference $|t_i - t_j|$ is less than or equal to T/M. Since $M > 2(N_1 + N_2)$, $|t_i - t_j|$ cannot be an integer multiple of T/N or $T/(2(N_1 + N_2))$.

See Appendix B-C for a proof.

The above observation leads to the following theorem for $M \leq N$:

[4]The reason for introducing this assumption is that we can always normalize the power gain of the filter by scaling the sampling noise power. More specifically, replacing some given $H(\omega)$ with $\alpha H(\omega)$ for some α, changes $\tilde{S}(t_i)$ to $\alpha\tilde{S}(t_i)$. Hence, $Y_i = \tilde{S}(t_i) + Z_i$ would change to $Y_i = \alpha\tilde{S}(t_i) + Z_i$. But reconstruction from Y_i, $(i = 1, 2, \ldots, M)$ yields the same distortion as the reconstruction from Y_i/α, which is equivalent to dividing the noise power by α.

Theorem 2: For $M \leq N$, the optimal average and variance of distortion are

$$\frac{\mathsf{D}_{\min}}{\mathsf{p}} = \frac{1}{2}\left(2N - M + \frac{M}{1+SNR}\right), \quad (11)$$

$$\frac{\mathsf{V}_{\min}}{\mathsf{p}^2} = 2N - M + \frac{M}{(1+SNR)^2}. \quad (12)$$

Both the minimal average and variance of the distortion are obtained by choosing M distinct time instances arbitrarily from the following set of N samples [5]

$$\{0, \frac{1}{N}T, \frac{2}{N}T, \frac{3}{N}T, \cdots, \frac{N-1}{N}T\}.$$

The optimal interpolation formula for this set of sampling points is given by

$$\hat{S}(t) = \frac{\mathsf{p}}{N\mathsf{p}+\sigma^2}\sum_{i=1}^{M}\sum_{\ell=N_1}^{N_2}\cos(\ell\omega_0(t-t_i))Y_i,$$
$$= \frac{\mathsf{p}}{N\mathsf{p}+\sigma^2}\sum_{i=1}^{M}\cos(\frac{\omega_0}{2}(t-t_i))\frac{\sin(\frac{N\omega_0}{2}(t-t_i))}{\sin(\frac{\omega_0}{2}(t-t_i))}Y_i, \quad (13)$$

where Y_i is the noisy sample of the signal at $t = t_i$.

See Appendix B-D for a proof. To understand the meaning of (11), observe that for fixed values of p and σ, the minimum distortion is linear in M. But it is not linear in N (as SNR in the denominator is $N\mathsf{p}/\sigma^2$), except when SNR goes to infinity. In the case of noiseless samples, $\sigma = 0$, the SNR will be infinity and the minimal distortion will be

$$\frac{\mathsf{p}}{2}(2N - M).$$

To intuitively understand this equation, observe that there are $2N$ free variables and we can recover M of them using the samples. Therefore, we will have $2N - M$ free variables, giving a total distortion of $(2N-M)\mathsf{p}/2$ as the power of each sinusoidal function is $\mathsf{p}/2$. Moreover, the maximum distortion is $2N(\mathsf{p}/2)$, which is obtained when $\sigma = \infty$ ($SNR = 0$) or $M = 0$. Finally, observe that when SNR is large,

$$\mathsf{D}_{\min} \approx \frac{\mathsf{p}}{2}\left(2N - M + \frac{M}{SNR}\right) = \frac{\mathsf{p}}{2}(2N-M) + \frac{M\sigma^2}{2N}$$

which is linear in both p and σ^2. Observe that $\mathsf{D}_{\min}$ is increasing in both p and σ^2, since $2N - M \geq 0$ for $M \leq N$. Furthermore, $(2N-M)/2 \geq M/(2N)$ when $M \leq N$, implying that the coefficient of p is greater than or equal to that of σ^2. Figure 2 depicts $\mathsf{D}_{\min}$ as a function of p, σ^2, M and N.

In the statement of Theorem 2, one possible optimal sampling set was given. Next, we consider the question of whether other optimal sampling sets exist for $M \leq N$:

Proposition 1: If $M \leq N$, and N does not divide $2N_1 - 1$, then a given set of sampling times $(t_1, t_2, \cdots, t_M)$ is optimal only if *there exists some constant $\tau < T/N$ such that for at least $M - 1$ of the sampling points t_i we have*

$$t_i \in \{\tau, \tau + \frac{1}{N}T, \tau + \frac{2}{N}T, \tau + \frac{3}{N}T, \cdots, \tau + \frac{N-1}{N}T\}.$$

See Appendix B-E for a proof.

B. Lower Bound 2 on Distortion

In the next lemma, alternative lower bounds on $\mathsf{D}_{\min}$ and $\mathsf{V}_{\min}$ are given, which are tighter than the ones given in Lemma 1 for $M > 2N$.

Lemma 2 (Lower Bound 2): For any $\sigma > 0$, the following lower bounds hold for all values of M, N and SNR:

$$\mathsf{D}_{\min} \geq \frac{N\mathsf{p}}{1 + \frac{M}{2N}SNR}, \quad (14)$$

$$\mathsf{V}_{\min} \geq \frac{2N\mathsf{p}^2}{(1 + \frac{M}{2N}SNR)^2}. \quad (15)$$

Furthermore, the above inequalities hold with equality for given values of M and N, if one can find a set of M distinct sampling times $\{t_1, t_2, \ldots, t_M\}$ such that

$$\sum_{i=1}^{M} e^{j2\pi k\frac{t_i}{T}} = 0 \quad for\ k \in \{1, 2, \cdots, N-1\} \times \cup\{2N_1, 2N_1+1, \cdots, 2N_2\}. \quad (16)$$

See Appendix B-F for a proof.

As an example, consider $M > 2N_2$ and uniform samples $t_i = (i-1)(T/M)$ for $i = 1, \ldots, M$. Then, using geometric series calculus, we have

$$\sum_{i=1}^{M} e^{j2\pi k\frac{i-1}{M}} = \frac{e^{j2\pi k\frac{M}{M}} - 1}{e^{j2\pi k\frac{1}{M}} - 1} = 0$$

for any natural number $1 \leq k \leq M - 1$. This shows that the bounds given above are tight via uniform sampling if $M > 2N_2$.

The above observation leads to the following theorem for $M \geq 2N$:

Theorem 3: The following lower bounds on optimal average and variance of distortion hold:

$$\frac{\mathsf{D}_{\min}}{\mathsf{p}} \geq \frac{N}{1 + \frac{M}{2N}SNR}, \quad (17)$$

$$\frac{\mathsf{V}_{\min}}{\mathsf{p}^2} \geq \frac{2N}{(1 + \frac{M}{2N}SNR)^2}. \quad (18)$$

Furthermore, for $M \geq 2N$, uniform sampling, i.e., *$t_i = iT/M$ ($i = 1, 2, \cdots, M$) is an optimal sampling strategy if no multiple of M can be found in the set $\{2N_1, 2N_1 + 1, \cdots, 2N_2\}$. In particular, uniform sampling is optimal when $M > 2N_2$. Also the reconstruction formula for the optimal set of sampling points is given by*

$$\hat{S}(t) = \frac{\mathsf{p}}{\frac{M}{2}\mathsf{p}+\sigma^2}\sum_{i=1}^{M}\sum_{\ell=N_1}^{N_2}\cos(\ell\omega_0(t-t_i))Y_i,$$
$$= \frac{\mathsf{p}}{N\mathsf{p}+\sigma^2}\sum_{i=1}^{M}\cos(\frac{\omega_0}{2}(t-t_i))\frac{\sin(\frac{N\omega_0}{2}(t-t_i))}{\sin(\frac{\omega_0}{2}(t-t_i))}Y_i, \quad (19)$$

where Y_i is the noisy sample of the signal at $t = t_i$.

[5] Observe that these points are uniform sampling points with increment T/N; they are not uniform sampling points corresponding to M samples, unless $M = N$. Uniform sampling for M points takes the samples at times $\{0, \frac{1}{M}T, \frac{2}{M}T, \frac{3}{M}T, \cdots, \frac{M-1}{M}T\}$.

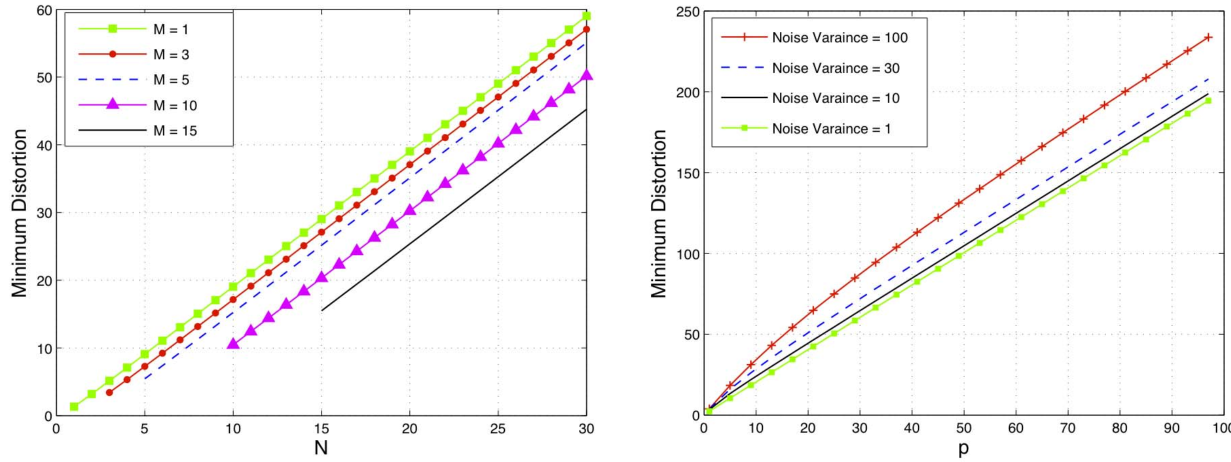


Fig. 2. In the left subfigure, the minimum average distortion is illustrated as a function of N for different values of $M \leq N$. In the right subfigure, the minimum average distortion is depicted as a function of p for different values of the noise variance σ^2.

See Appendix B-G for a proof. The condition given for the optimality of uniform sampling comes from (16) in the statement of Lemma 2. Unlike the case of $M \leq N$, where the optimal sampling points could be found without any need to know N_1, observe that the constraint that no multiple of M can be found in the set $\{2N_1, 2N_1+1, \cdots, 2N_2\}$ depends on N_1. In fact, the uniform sampling strategy, $t_i = iT/M$, is not in general optimal. For instance, uniform sampling is not an optimal strategy if there is some natural number $k \in \{1, 2, \cdots, N-1\} \cup \{2N_1, 2N_1+1, \cdots, 2N_2\}$ that is a multiple of M, since in this case (16) becomes $\sum_{i=1}^{M} e^{j2\pi k \frac{t_i}{T}} = \sum_{i=1}^{M} e^{j2\pi k \frac{i}{M}} = M \neq 0$. In fact, non-uniform sampling may be optimal in some cases. Below are some examples of non-uniform sampling sets that are the solutions of (16):

Example 1: When $M = 2$, $N = 1$ and $N_1 = 1$, sampling points $t_1 = 0, t_2 = T/4$ are optimal.

Example 2: Given any N_1 and N_2, let $\mathcal{K} = \{1, 2, \cdots, N-1\} \cup \{2N_1, 2N_1+1, \cdots, 2N_2\}$. Observe that the sets $\{1, 2, \cdots, N-1\}$ and $\{2N_1, 2N_1+1, \cdots, 2N_2\}$ might have an intersection. Let $\nu = |\mathcal{K}| \leq N-1+2N_2-2N_1+1$. Let us index the elements of $\mathcal{K}$ as $\{k_1, \cdots, k_\nu\}$ in some arbitrary order. We introduce the following $M = 2^\nu$ non-uniform sampling points, and claim that it is an optimal sampling set. For any $i \in [0 : M-1]$, let

$$t_i = \sum_{r=1}^{\nu} b_r \frac{T}{2k_r},$$

where $(b_1, \cdots, b_\nu) \in \{0,1\}^\nu$ is the binary expansion of time index $i \in [0 : M-1]$. We verify that (16) holds:

$$\sum_i e^{j2\pi k \frac{t_i}{T}} = \sum_{b_1, \cdots, b_\nu \in \{0,1\}} e^{j2\pi k (\sum_{r=1}^{\nu} b_r \frac{1}{2k_r})} \tag{20}$$

$$= \prod_{r=1}^{\nu} \left(\sum_{b_r} e^{j\pi b_r \frac{k}{k_r}} \right) \tag{21}$$

$$= \prod_{r=1}^{\nu} \left(1 + e^{j\pi \frac{k}{k_r}} \right), \tag{22}$$

which is zero if $k \in \mathcal{K}$, i.e., *$k = k_r$ for some r.*

C. Sampling Distortion Tradeoffs for $N < M \leq 2N$

Given an arbitrary $M \leq 2N$, Lower bound 1 (Lemma 1) is tight when N divides $2N_1 - 1$, *i.e.,* $N | 2N_1 - 1$. This is because $N | 2N_1 - 1$ implies that N divides $2N_1 - 1 + N = N_1 + N_2$, and using conditions (9) and (10) from the statement of Lemma 1, any subset of size M of

$$\Big\{0, \frac{1}{N}T, \frac{2}{N}T, \cdots, \frac{N-1}{N}T, \frac{T}{2(N_1+N_2)}, \frac{1}{N}T + \frac{T}{2(N_1+N_2)}, \cdots, \frac{N-1}{N}T + \frac{T}{2(N_1+N_2)}\Big\},$$

is an optimal sampling set and thus Lower bound 1 holds with equality.

However, when N does not divide $2N_1 - 1$, one cannot choose the sampling times so that the equality conditions given in the two general lower bounds are met for $N < M \leq 2N$.

Any particular sampling strategy would give us an upper bound on $\mathsf{D}_{\min}$. One sampling strategy is to take an optimal sampling strategy for $M = N$, and append it by adding $M - N$ extra sampling times. More specifically, we choose the sampling set

$$\{t_1, t_2, \ldots, t_M\} = \{0, \frac{1}{N}T, \frac{2}{N}T, \frac{3}{N}T, \cdots, \frac{N-1}{N}T, \times \frac{1}{2N}T, \frac{3}{2N}T, \cdots, \frac{2M-2N-1}{2N}T\}. \tag{23}$$

This leads to the following upper bound on the average distortion.

Theorem 4 (Upper Bound): For $N < M \leq 2N$, the optimal average distortion can be bounded as follows

$$\frac{\mathsf{D}_{\min}}{\mathsf{p}} \leq \frac{1}{2}(2N - M) + \frac{2N - M}{2(1+SNR)} + \mathsf{Num} \cdot \frac{1+SNR}{1+2\ SNR} + (M - N - \mathsf{Num}) \cdot \frac{1}{1+SNR}, \tag{24}$$

in which

$$\mathsf{Num} = \min\left(f(N_1, N), M - N\right)$$

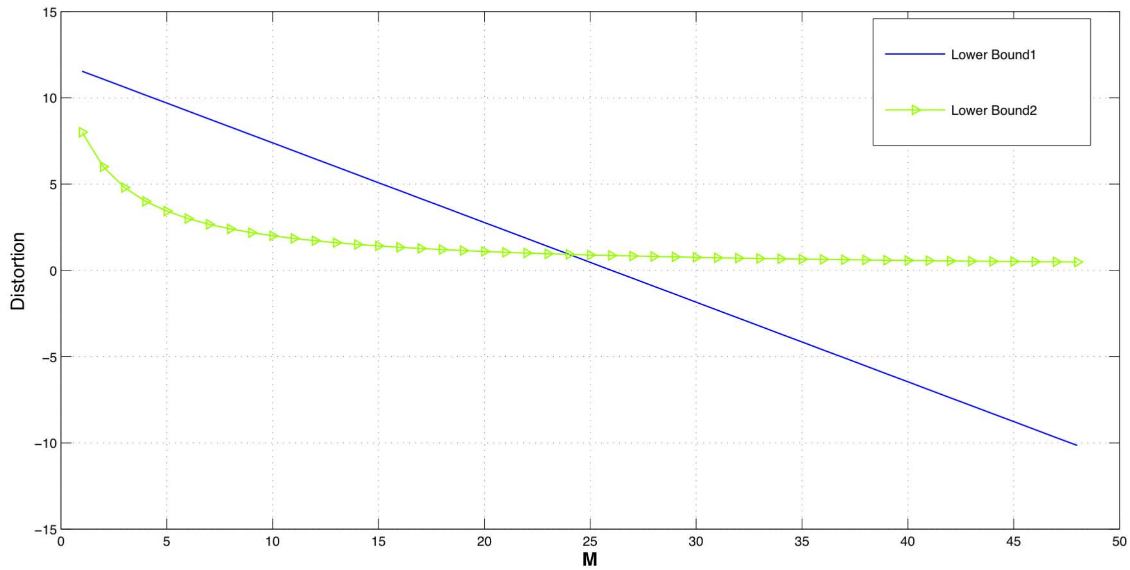


Fig. 3. The lower bounds on the average distortion given in Lemmas 1 and 2.

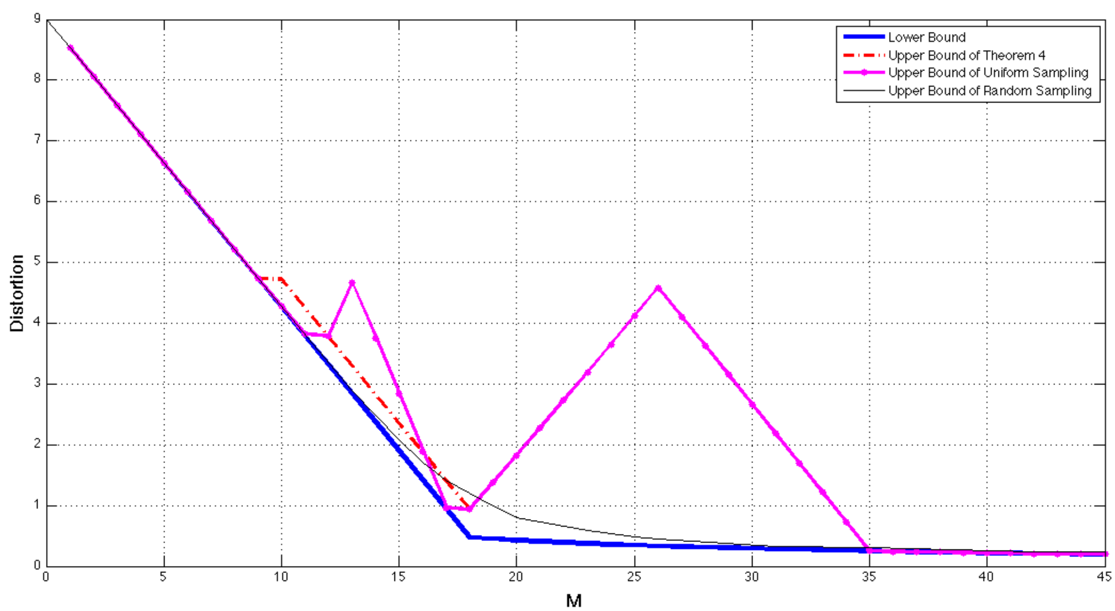


Fig. 4. The average distortion for $(N_1, N, \mathsf{p}, \sigma) = (9, 9, 1, 0.5)$. The signal period is $T = 1$. Here M is shown on the x-axis. The union lower bound (blue stretched-line curve) and the upper bound given in Theorem 4 (red dashed curve) for M between N and $2N$ are depicted. Moreover, the two upper bounds of uniform and random sampling are are also plotted (purple dotted-line and black curves, respectively).

and $f(a, b)$ is equal to

$$f(a,b) = \begin{cases} b-1 & \text{if } r = 0 \\ 2b - 2r + 1 & \text{if } 2r > b \\ 2r - 1 & \text{if } 0 < 2r \leq b, \end{cases}$$

where r is the remainder of dividing a by b.

See Appendix B-H for a proof.

D. Plots of the Lower and Upper Bounds

In Fig. 3, the two lower bounds on the optimal average distortion $\mathsf{D}_{\min}$ derived in Lemmas 1 and 2 are plotted as a function of number of samples M, where we have fixed all the other parameters such as N and σ^2. Lower bound 1 is tight and equal to $\mathsf{D}_{\min}$ for $M \leq N$, as shown in Theorem 2, and Lower bound 2 is tight when M is large, for $M > 2N_2$, as shown in Theorem 3.

The maximum of the two lower bounds (Lower bound 1 and Lower bound 2) is plotted with blue stretched-line in Fig. 4. This curve matches $\mathsf{D}_{\min}$ for $M \leq N = 9$ and $M > 2N_2 = 34$. In this figure, the distortion of uniform sampling is also depicted (purple dotted-line curve), which constitutes an upper bound on the optimal minimum average distortion $\mathsf{D}_{\min}$. The upper bound of Theorem 4 for $N = 9 \leq M \leq 2N = 18$ is the

$$Q = \begin{pmatrix} \cos(N_1\omega_0 t_1) & \cos((N_1+1)\omega_0 t_1) & \cdots & \cos(N_2\omega_0 t_1) & \sin(N_1\omega_0 t_1) & \sin((N_1+1)\omega_0 t_1) & \cdots & \sin(N_2\omega_0 t_1) \\ \cos(N_1\omega_0 t_2) & \cos((N_1+1)\omega_0 t_2) & \cdots & \cos(N_2\omega_0 t_2) & \sin(N_1\omega_0 t_2) & \sin((N_1+1)\omega_0 t_2) & \cdots & \sin(N_2\omega_0 t_2) \\ & & & \vdots & \vdots & & & \\ \cos(N_1\omega_0 t_M) & \cos((N_1+1)\omega_0 t_M) & \cdots & \cos(N_2\omega_0 t_M) & \sin(N_1\omega_0 t_M) & \sin((N_1+1)\omega_0 t_M) & \cdots & \sin(N_2\omega_0 t_M) \end{pmatrix}$$

red dashed curve. Observe that uniform sampling distortion curve matches the lower bound for for $M > 2N_2 = 34$; this is consistent with Theorem 3. Moreover, note that the distortion of uniform sampling is not monotonically decreasing in the sampling frequency (this is unlike the performance of nonuniform sampling which is decreasing in the number of samples). Finally, the performance of the uniform sampling is close to optimal for $M \leq N = 8$ (its curve almost matches that of the lower bound in Lemma 1 which is optimal for $M \leq N$). More specifically, there is only a percentage gain of about 0.3 by using the optimal non-uniform sampling strategy. However, a main advantage of the optimal nonuniform sampling is its *robustness* with respect to missing samples (as discussed in the introduction). Finding the optimal sampling locations is a computationally cumbersome problem when the number of sampling points M is large. One search strategy is to utilize a random search, *e.g.,* to repeatedly and randomly choose the time instances $\{t_i\}$, and output the best time instances. The output of this algorithm will serve as an upper bound on the optimal MMSE average distortion. This upper bound is also depicted in this figure (black curve).

In Fig. 5, we plot a sample signal, its MMSE reconstruction as well as 95 percent confidence intervals (2 times the standard deviation of $S(t)$) for two values of noise variance. To compute the confidence intervals, we need to compute the distribution of $S(t)$ given the observation $\mathbf{Y} = \mathbf{y}$. This conditional distribution is a Gaussian distribution because the vector of Fourier coefficients $\mathbf{X}$ and $\mathbf{Y}$ are jointly Gaussian, and $S(t)$ is linear in the entries of $\mathbf{X}$. The expected value of $S(t)$ given $\mathbf{Y} = \mathbf{y}$ is our MMSE reconstruction. Its variance of $S(t)$ given $\mathbf{Y} = \mathbf{y}$ does not depend on the particular value $\mathbf{y}$ and is determined by the covariance matrix of the reconstruction error $\mathbf{X} - \hat{\mathbf{X}} = \mathbf{X} - \mathbb{E}[\mathbf{X}|\mathbf{Y}]$.

IV. The Problem in Matrix Form and Proof Techniques

A. Problem Representation in a Matrix Form

Let $\mathbf{X}$ be the vector of coefficients A_ℓ and B_ℓ ($\ell = N_1, N_1+1, \cdots, N_2$) of the original signal, $S(t)$, given in (2), *i.e.,*

$$\mathbf{X} = [A_{N_1}, \cdots, A_{N_2}, B_{N_1}, \cdots, B_{N_2}]^\dagger. \tag{25}$$

Since the vector $\mathbf{X}$ consists of real numbers, † is just the transpose operation in the above equation. After filtering $S(t)$, we get

$$\tilde{S}(t) = \sum_{\ell=N_1}^{N_2} [\tilde{A}_\ell \cos(\ell\omega_0 t) + \tilde{B}_\ell \sin(\ell\omega_0 t)] \tag{26}$$

in which

$$\begin{cases} \tilde{A}_\ell = A_\ell H_R(\ell\omega_0) + B_\ell H_I(\ell\omega_0), \\ \tilde{B}_\ell = B_\ell H_R(\ell\omega_0) - A_\ell H_I(\ell\omega_0), \end{cases} \tag{27}$$

for $\ell = N_1, N_1+1, \cdots, N_2$, where $H_R(\omega)$ and $H_I(\omega)$ are the real and imaginary parts of $H(\omega)$, respectively. If we denote the vector of coefficients of $\tilde{S}(t)$ as

$$\tilde{\mathbf{X}} = [\tilde{A}_{N_1}, \cdots, \tilde{A}_{N_2}, \tilde{B}_{N_1}, \cdots, \tilde{B}_{N_2}]^\dagger$$

and express (27) in a matrix form, we get

$$\tilde{\mathbf{X}} = L\mathbf{X},$$

where L is of the following form

$$L = \begin{pmatrix} L_1 & L_2 \\ -L_2 & L_1 \end{pmatrix}, \tag{28}$$

in which

$$L_1 = \begin{pmatrix} H_R(N_1\omega_0) & 0 & \cdots & 0 \\ 0 & H_R((N_1+1)\omega_0) & 0 & 0 \\ & \vdots & & \\ 0 & \cdots & 0 & H_R(N_2\omega_0) \end{pmatrix}$$

$$L_2 = \begin{pmatrix} H_I(N_1\omega_0) & 0 & \cdots & 0 \\ 0 & H_I((N_1+1)\omega_0) & 0 & 0 \\ & \vdots & & \\ 0 & \cdots & 0 & H_I(N_2\omega_0) \end{pmatrix}.$$

The signal $\tilde{S}(t)$ is then sampled at time instances t_i for $i = 1, 2 \cdots, M$. We have

$$\tilde{S}(t_i) = \sum_{\ell=N_1}^{N_2} [\tilde{A}_\ell \cos(\ell\omega_0 t_i) + \tilde{B}_\ell \sin(\ell\omega_0 t_i)].$$

Let $\tilde{\mathbf{S}}$ be the vector of the samples

$$\tilde{\mathbf{S}} = [\tilde{S}(t_1), \tilde{S}(t_2), \cdots, \tilde{S}(t_M)]^\dagger.$$

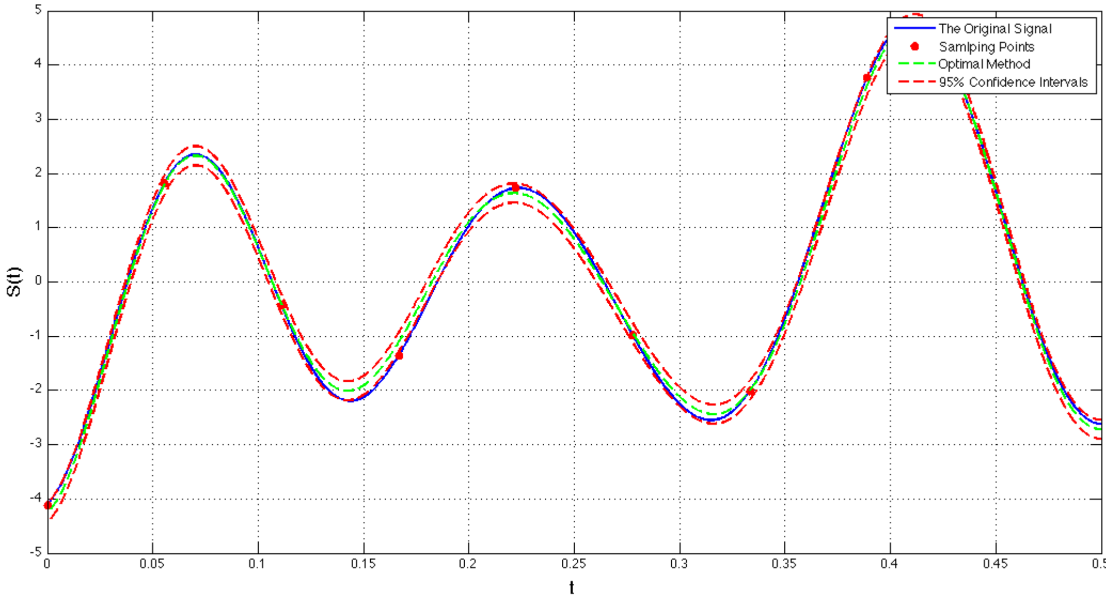
Therefore, in the matrix form we have $\tilde{\mathbf{S}} = Q\tilde{\mathbf{X}} = QL\mathbf{X}$, where Q is an $M \times 2N$ matrix given on top of this page. Finally, the vector of observations denoted by $\mathbf{Y} = \tilde{\mathbf{S}} + \mathbf{Z}$ can be written as

$$\mathbf{Y} = QL\mathbf{X} + \mathbf{Z},$$

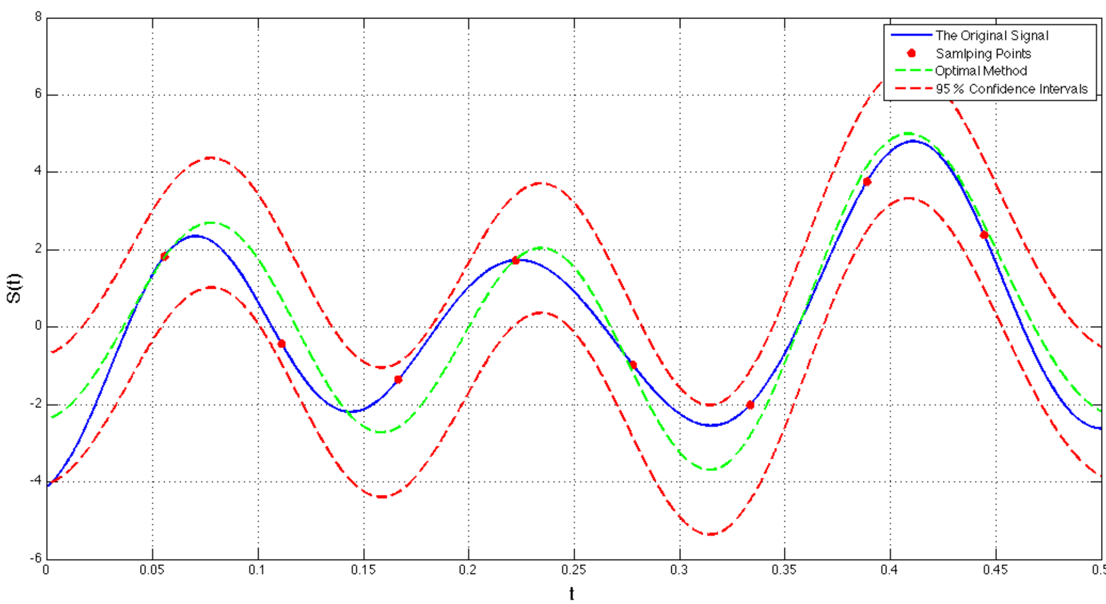
where $\mathbf{Z}$ is the noise vector.

Estimation of $S(t)$ is equivalent to the estimation of its Fourier coefficients. If we denote the coefficients of the reconstruction signal by

$$\hat{\mathbf{X}} = [\hat{A}_{N_1}, \cdots, \hat{A}_{N_2}, \hat{B}_{N_1}, \cdots, \hat{B}_{N_2}]^\dagger,$$



(a)



(b)

Fig. 5. Random signals generated with parameters $(N, N_1, N_2, \mathsf{p}) = (7, 2, 8, 1)$ for different values of noise variances $\sigma^2 = 0.01$ and 1 are depicted in color blue in subfigures (*a*) and (*b*), respectively. Their optimal MMSE reconstructions with $M = 18$ samples is given in color green (dash line curve). Additionally, we plot two lower and upper curves which are the 95 percent confidence intervals for the MMSE reconstruction in color red (dash-line curve).

using the Parseval's theorem, the sampling distortion is equal to

$$\begin{aligned}\frac{1}{T}\int_0^T |\hat{S}(t) - S(t)|^2 dt &= \frac{1}{2}\sum_{\ell=N_1}^{N_2} (|\hat{B}_\ell - B_\ell|^2 + |\hat{A}_\ell - A_\ell|^2) \\ &= \frac{1}{2}\|\mathbf{X} - \hat{\mathbf{X}}\|^2, \end{aligned} \tag{29}$$

which is a random variable. Our goal is to minimize its average and variance.

B. Overview of Proof Techniques

In this section, we provide some intuitions about the proofs of our main results; the formal proofs are given in the appendices. To reconstruct the original signal $S(t)$ given in (2), one should estimate its Fourier coefficients A_ℓ and B_ℓ $(\ell = N_1, N_1 + 1, \cdots, N_2)$. Since all the random variables are

Gaussian, the linear MMSE is optimal and thus we would like to use $QL\mathbf{X}+\mathbf{Z}$ to find $\hat{\mathbf{X}}$ such that $E\|\mathbf{X}-\hat{\mathbf{X}}\|^2$ is minimized. There are two ways to express the mean square error; see [21, p. 90]. These two formulas reduce to the following (as formally shown in Section V-A):

$$\begin{aligned}\mathbb{E}\|\mathbf{X}-\hat{\mathbf{X}}\|^2 &= (2N-M)\mathsf{p}+\mathsf{p}\sigma^2\mathtt{Tr}[(\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2 I)^{-1}] \quad (30)\\ &= \mathsf{p}\sigma^2\mathtt{Tr}[(\mathsf{p}L^\dagger Q^\dagger QL+\sigma^2 I)^{-1}]. \quad (31)\end{aligned}$$

We first minimize the above expressions over the matrix L for a given matrix Q. We give the details in the proof of Theorem 1, but a main step is to show that $\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2 I\le \mathsf{p}QQ^\dagger+\sigma^2 I$, and then show that $L=I$ minimizes (31). The main difficulty is minimization over the matrix Q. The difficulty is that we need to minimize the trace of a function of the matrix Q which has coordinates that depend on the sampling times t_i in a non-linear way.

Minimizing Over the Matrix Q: When we do not use a pre-sampling filter, the two MMSE formulas given in (30) and (31) reduce to

$$\begin{aligned}\mathbb{E}\|\mathbf{X}-\hat{\mathbf{X}}\|^2 &= (2N-M)\mathsf{p}+\mathsf{p}\sigma^2\mathtt{Tr}[(\mathsf{p}QQ^\dagger+\sigma^2 I)^{-1}] \quad (32)\\ &= \mathsf{p}\sigma^2\mathtt{Tr}[(\mathsf{p}Q^\dagger Q+\sigma^2 I)^{-1}]. \quad (33)\end{aligned}$$

Next, we use the following inequality from majorization theory: Let Δ be a positive definite matrix, and Δ_{diag} be a diagonal matrix constructed by keeping the diagonal entries of Δ and setting the off-diagonal entries to be zero, *i.e.,* the (i,j)th entry of Δ_{diag} is zero if $i\neq j$, and the (i,i)th entry of Δ_{diag} is the (i,i)th entry of Δ. Then we have the following inequality:

$$\mathtt{Tr}(\Delta^{-1})\ge\mathtt{Tr}(\Delta_{\mathsf{diag}}^{-1}).$$

Alternatively, because Δ_{diag} is a diagonal matrix, the inequality says that

$$\mathtt{Tr}(\Delta^{-1})\ge\sum_i\frac{1}{\Delta_{(i,i)}}. \quad (34)$$

Furthermore, the above inequality is strict if Δ has at least one non-zero off-diagonal entry. In other words, equality holds if and only if Δ is a diagonal matrix ($\Delta_{\mathsf{diag}}=\Delta$). In fact, more generally from [8, Ex. II.1.12 and Th. II.3.1], we have the following theorem:

Theorem 5: For any convex function $f(\cdot)$ and any real symmetric matrix Δ, we have

$$\mathtt{Tr}(f(\Delta))\ge\mathtt{Tr}(f(\Delta_{\mathsf{diag}}))=\sum_i f(\Delta_{(i,i)}),$$

where a function of a matrix, $f(\Delta)$, is defined as follows: for a diagonal matrix Λ, $f(\Lambda)$ is defined by applying $f(\cdot)$ to all the diagonal entries. For a symmetric real matrix Δ, $f(\Delta)$ is defined as $Pf(\Lambda)P^{-1}$ if $\Delta=P\Lambda P^{-1}$ for some diagonal matrix Λ.

If $f(\cdot)$ is strictly convex, i.e., *$f''(x)>0$, then equality in the above relation holds if and only if Δ is a diagonal matrix.*

Now, by setting $\Delta_1=\mathsf{p}QQ^\dagger+\sigma^2 I$ and using the above trace inequality (34) in equation (32), we get a lower bound on the reconstruction distortion:

$$\begin{aligned}\mathbb{E}\|\mathbf{X}-\hat{\mathbf{X}}\|^2 &= (2N-M)\mathsf{p}+\mathsf{p}\sigma^2\mathtt{Tr}\left(\Delta_1^{-1}\right) \quad (35)\\ &\ge (2N-M)\mathsf{p}+\mathsf{p}\sigma^2\sum_i\left(\Delta_{1(i,i)}\right)^{-1}. \quad (36)\end{aligned}$$

The above formula leads to what we call the *Lower bound 1* on distortion. Similarly, by setting $\Delta_2=\mathsf{p}Q^\dagger Q+\sigma^2 I$ and using the above trace inequality (34) in equation (33), we get another lower bound on the reconstruction distortion. This leads to what we call the *Lower bound 2* on distortion.

The next question is whether any of these two lower bounds can be tight. The trace inequality (34) is tight if and only if Δ is diagonal. Then, the question is whether we can choose the values of sampling times t_i such that either $\Delta_1=\mathsf{p}QQ^\dagger+\sigma^2 I$ or $\Delta_2=\mathsf{p}Q^\dagger Q+\sigma^2 I$ become diagonal matrices. Each off-diagonal entry of Δ_1 or Δ_2 is a function of the sampling times t_i. Setting an off-diagonal entry to zero imposes an equation on t_i. The question is whether we can simultaneously solve all of these (non-linear) equations corresponding to all the off-diagonal entries. It turns out that if the number of samples M is small (less than or equal to N), then we appropriately choose t_i (in a nonuniform manner in the $[0,T]$ interval) such that Δ_1 becomes diagonal (Theorem 2). If M is large (greater than $2N$), then under some constraints we can make Δ_2 diagonal; when M is very large (greater than $2N_2$), the uniform sampling strategy always makes Δ_2 diagonal (Theorem 3). Unfortunately, when M is in the intermediate range, between N and $2N$, we cannot make either Δ_1 or Δ_2 become diagonal with appropriate choice of t_i. Nonetheless, backed by numerical simulations, one possible strategy is to choose the t_i in such a way that forces a good fraction (if not all) of the entries of Δ_1 to become zero. More specifically, we use the sampling set given in (23). While it is not necessarily true that this approach for choosing t_i is optimal, nevertheless the reconstruction distortion incurred by it provides an upper bound on the optimal reconstruction distortion (Theorem 4). As it becomes clear from the proof, this makes $\Delta_1=\mathsf{p}QQ^\dagger+\sigma^2 I$ having the following block form (c is a constant): $\Delta_1=c\begin{bmatrix}I & \Gamma\\ \Gamma^\dagger & I\end{bmatrix}$ with two identity blocks appearing on the diagonal. Thus, while this choice cannot make Δ_1 diagonal (the off-diagonal block Γ has non-zero entries), it produces two identity matrices on the main block. The distortion depends on the inverse of Δ_1 (see (35)). While it is possible to relate eigenvalues of Δ_1 to singular values of Γ via the Jordan–Wielandt theorem (Theorem 9), it is not easy to find an explicit expression for the singular values of Γ. This is due to the fact that the entries of Γ are summations of some sine and cosine terms and do not have a clean expression. Our idea is to expand the matrix Γ by adding new rows or columns, such that the singular values of the extended matrix can be explicitly calculated. Next, we use results from linear algebra that relate singular values of a matrix to the singular values of its submatrices. While the expression of the upper bound on distortion looks complicated (due to the fact that various linear algebra tools are invoked), but we emphasize that the given expression is nonetheless *explicit* and can be used for analytical considerations.

V. Remarks and Extensions

A. Suboptimality of Filtering at Half the Sampling Rate

Consider the case of $M = N$. According to Theorem 2, the uniform sampling set

$$\{t_1, t_2, \cdots, t_M\} = \{0, \frac{1}{N}T, \frac{2}{N}T, \frac{3}{N}T, \cdots, \frac{N-1}{N}T\} \quad (37)$$

is optimal and obtains the minimum distortion

$$\mathsf{D}_{\min} = \frac{\mathsf{p}}{2}\left(M + \frac{M}{1+SNR}\right).$$

Here, it is assumed that no pre-sampling filter is used (following from the statement of Theorem 1). Observe that the signal bandwidth, Nf_0 is equal to the sampling rate Mf_0. Here the sampling rate is *not* twice the signal bandwidth. Suppose we use the uniform sampling points given in (37), and employ a pre-sampling filter that reduces the signal bandwidth to *half* the sampling rate as follows (an anti-aliasing filter):

$$H(\ell\omega_0) = \begin{cases} 1, & N_1 \le \ell < N_1 + M/2 \\ 0, & \text{otherwise} \end{cases}. \quad (38)$$

Here the number of samples, M, is assumed to be even. This filter makes all of the coefficients A_ℓ and B_ℓ equal to zero, except for M variables A_ℓ, B_ℓ when $N_1 \le \ell < M/2 + N_1$. We then have the following theorem:

Theorem 6: Suppose that we insist on using uniform sampling at rate M, i.e., *taking the points*

$$\{t_1, t_2, t_3, \cdots, t_M\} = \{0, \frac{1}{M}T, \frac{2}{M}T, \frac{3}{M}T, \cdots, \frac{M-1}{M}T\}.$$

Then, the performance of the filter H satisfies:

$$\mathsf{D}_H > \frac{\mathsf{p}}{2}\left(M + \frac{M}{1+\frac{1}{2}SNR}\right), \quad (39)$$

which is strictly greater than

$$\mathsf{D}_{\min} = \frac{\mathsf{p}}{2}\left(M + \frac{M}{1+SNR}\right),$$

even when we increase the signal power by a factor of two.

See Appendix B-I for a proof.

Discussion 1: Anti-aliasing filter H is analogous to an *interference cancellation* scheme when we interpret aliasing as interference. But without a filter, we can do *interference management* and use the interfered aliased information to recover the signal. To convey the essential intuition, consider the following different but simpler problem: suppose A, $\mathsf{B} \sim \mathcal{N}(0, \mathsf{p})$ are two independent Gaussian random variables. We are interested in recovering these two variables via one observation that is corrupted by a Gaussian noise of variance σ^2. The interference cancellation strategy corresponds to discarding B and observing $\mathsf{A} + Z$, where $Z \sim \mathcal{N}(0, \sigma^2)$ is the noise. This yields an estimation error of

$$\mathsf{p} + \frac{\mathsf{p}\sigma^2}{(\mathsf{p}+\sigma^2)} = \mathsf{p}(1 + \frac{1}{1+SNR/2}), \quad (40)$$

where the p and $\mathsf{p}\sigma^2/(\mathsf{p}+\sigma^2)$ are the estimation errors for B and A, respectively, and SNR is defined as $2\mathsf{p}/\sigma^2$. On the other hand, an interference management scheme observes $\mathsf{A} + \mathsf{B} + Z$. Here both A and B are interference terms for each other. The total estimation error in this case is equal to

$$2\frac{\mathsf{p}(\mathsf{p}+\sigma^2)}{2\mathsf{p}+\sigma^2} = \mathsf{p}(1 + \frac{1}{1+SNR}). \quad (41)$$

In comparison to (40), the SNR gain of two is attained due to the interference management.

B. Extension to Arbitrary Power Values

Consider an extension of the results to the case when the coefficients A_ℓ and B_ℓ in (2) are mutually independent zero-mean Gaussian r.v.s with arbitrary variances p_ℓ for $N_1 \le \ell \le N_2$. Observe that this extension includes multi-band signals as well as sparse signals with known frequency support (when p_ℓ is non-negligible for few values of ℓ).

For the case of positive equal variances $\mathsf{p}_\ell = \mathsf{p} > 0$ for $\ell = N_1, \cdots, N_2$, we have provided two general lower bounds on the average distortion. When p_ℓ are arbitrary, we can generalize the second lower bound given in Lemma 2 as follows:

Proposition 2: For any $\sigma > 0$, the following lower bound on the average distortion holds for all values of M:

$$\mathsf{D}_{\min} \ge \sum_{\ell=N_1}^{N_2} \frac{1}{\frac{1}{\mathsf{p}_\ell} + \frac{M}{2\sigma^2}}. \quad (42)$$

The above inequalities are tight for some values of M and N. The condition for equality is the possibility of choosing the sampling time instances to satisfy the following equation:

$$\sum_{i=1}^{M} e^{j2\pi k\frac{t_i}{T}} = 0 \quad \text{for} \quad k \in \{1, 2, \cdots, N-1\} \cup \{2N_1, 2N_1+1, \cdots, 2N_2\}. \quad (43)$$

Uniform sampling, i.e., *$t_i = iT/M, i = 1, 2, \cdots, M$ is a solution to the above equation if for each k in the interval $2N_1 \le k \le 2N_2$, M does not divide k. In particular, for the rates above the Nyquist rate,* i.e., *$M > 2N_2$, uniform sampling is optimal.*

See Appendix B-J for a proof.

Appendix A
Estimator for Minimizing Variance of Distortion

The conventional MMSE estimation problem for estimating a vector $\mathbf{X}$ from vector $\mathbf{Y}$ asks for minimizing the expected value of the distortion $\|\mathbf{X} - \hat{\mathbf{X}}\|^2$ where the estimator $\hat{\mathbf{X}}$ is created as a function of observation $\mathbf{Y}$. However, this would only ensure that the distortion is minimized on *average*. In practice, we get one copy of $\mathbf{X}$ and $\mathbf{Y}$, and we want to ensure that the distortion that we obtain is small for that one copy, not just on average. Minimizing the variance of $\|\mathbf{X} - \hat{\mathbf{X}}\|^2$ makes sense because variance is a measure of concentration around the mean. For instance, by Chebyshev's inequality, the probability of $\|\mathbf{X} - \hat{\mathbf{X}}\|^2$ exceeding a threshold depends both on the average of $\|\mathbf{X} - \hat{\mathbf{X}}\|^2$ and its variance.

In this Appendix we show that for jointly Gaussian random variables, the estimator that minimizes the expected value of $\|\mathbf{X}-\hat{\mathbf{X}}\|^2$, also minimizes its variance. More specifically, we know that $\hat{\mathbf{X}}=\mathbb{E}[\mathbf{X}|\mathbf{Y}]$ minimizes $\mathbb{E}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2)$. We show that for *jointly Gaussian vectors*, $\hat{\mathbf{X}}=\mathbb{E}[\mathbf{X}|\mathbf{Y}]$ also minimizes the *difference*

$$\mathsf{Var}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2)=\mathbb{E}(\|\mathbf{X}-\hat{\mathbf{X}}\|^4)-\big(\mathbb{E}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2)\big)^2.$$

Theorem 7: Suppose $\mathbf{X}$ *and* $\mathbf{Y}$ *are two correlated jointly Gaussian vectors, having covariance matrices* C_X, C_Y *and* C_{XY}. *Let* $\hat{\mathbf{X}}$ *be a function of* $\mathbf{Y}$, *and consider the cost*

$$\mathsf{Var}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2).$$

The estimator $\hat{\mathbf{X}}$ *that minimizes the above cost constraint is* $\hat{\mathbf{X}}=\mathbb{E}[\mathbf{X}|\mathbf{Y}]$, *and the minimum variance is equal to* $2\mathtt{Tr}(C_e^2)$, *where*

$$C_e=C_X-C_{XY}C_{\mathbf{Y}}^{-1}C_{YX}.$$

Proof: We have

$$\mathsf{Var}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2)=\mathbb{E}_{\mathbf{Y}}\mathsf{Var}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2|\mathbf{Y})+\mathsf{Var}_{\mathbf{Y}}[\mathbb{E}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2|\mathbf{Y})]. \quad (44)$$

We claim that both of the terms $\mathbb{E}_{\mathbf{Y}}\mathsf{Var}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2|\mathbf{Y})$ and $\mathsf{Var}_{\mathbf{Y}}[\mathbb{E}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2|\mathbf{Y})]$ are minimized when $\hat{\mathbf{X}}=\mathbb{E}[\mathbf{X}|\mathbf{Y}]$. The second term is always non-negative and becomes zero when $\hat{\mathbf{X}}=\mathbb{E}[\mathbf{X}|\mathbf{Y}]$. This is because $\mathbb{E}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2|\mathbf{Y})$ will be equal to $\mathsf{Var}(\mathbf{X}|\mathbf{Y})$ which is a constant and does not depend on the value of $\mathbf{Y}$ for jointly Gaussian random variables. Therefore, its variance is zero if $\hat{\mathbf{X}}=\mathbb{E}[\mathbf{X}|\mathbf{Y}]$.

To prove that the first term is minimized at $\hat{\mathbf{X}}=\mathbb{E}[\mathbf{X}|\mathbf{Y}]$, it suffices to show that this claim for all values of $\mathbf{y}$. We will be done if we have the following statement: for any jointly Gaussian vector $\mathbf{X}$, the function

$$f(\mathbf{c})=\mathsf{Var}(\|\mathbf{X}-\mathbf{c}\|^2)$$

is minimized at $\mathbf{c}=\mathbb{E}[\mathbf{X}]$. Let $\mathbf{X}\sim\mathcal{N}(\mu,C_{\mathbf{X}})$. Then observe that $\mathbf{X}-\mathbf{c}\sim\mathcal{N}(\mu-\mathbf{c},C_{\mathbf{X}})$. If we replace $\mathbf{X}-\mathbf{c}$ by $U(\mathbf{X}-\mathbf{c})$ for some unitary matrix U, the norm $\|\mathbf{X}-\mathbf{c}\|^2$ remains invariant. Therefore, without loss of generality we can assume that $C_{\mathbf{X}}$ is diagonal. Then $\|\mathbf{X}-\mathbf{c}\|^2=\sum_i^n(X_i-c_i)^2$ is the sum of independent Gaussian random variables. Let σ_i^2 be the variance of X_i. Therefore

$$\begin{aligned}
f(\mathbf{c})&=\sum_i \mathsf{Var}((X_i-c_i)^2) && (45)\\
&=\sum_i \mathbb{E}((X_i-c_i)^4)-\mathbb{E}((X_i-c_i)^2)^2 && (46)\\
&=\sum_i \big[(\mu_i-c_i)^4+6(\mu_i-c_i)^2\sigma_i^2\\
&\qquad +3\sigma_i^4-(\sigma_i^2+(\mu_i-c_i)^2)^2\big] && (47)\\
&=\sum_i 4(\mu_i-c_i)^2\sigma_i^2+2\sigma_i^4, && (48)
\end{aligned}$$

which is clearly minimized when $c_i=\mu_i$, and the minimum will be equal to $2\sum_i\sigma_i^4=2\mathtt{Tr}(C_{\mathbf{X}}^2)$.

Coming back to the original minimization problem, we see that the covariance matrix of $\mathbf{X}$ given any value for $\mathbf{Y}=\mathbf{y}$, does not depend on the value of $\mathbf{y}$, and is equal to C_e, where $C_e=C_{\mathbf{X}}-C_{\mathbf{XY}}C_{\mathbf{Y}}^{-1}C_{\mathbf{YX}}$. Therefore, the overall minimum variance is equal to $2\mathtt{Tr}(C_e^2)$. ■

Appendix B
Proofs

Before giving the proofs, we first provide formulas for the average and variance of distortion in Section B-A.

A. Computing the Average and Variance of Distortion

Remember that the sampling distortion was found in (29) to be equal to

$$\begin{aligned}
\frac{1}{T}\int_0^T|\hat{S}(t)-S(t)|^2dt&=\frac{1}{2}\sum_{\ell=N_1}^{N_2}|\hat{B}_\ell-B_\ell|^2+|\hat{A}_\ell-A_\ell|^2\\
&=\frac{1}{2}\|\mathbf{X}-\hat{\mathbf{X}}\|^2, \qquad (49)
\end{aligned}$$

which is a random variable. Our goal is to minimize its average and variance.

B. Computing Average Distortion

We use $\mathbf{Y}$ to estimate the signal with minimal distortion. In other words, from (29), we would like to minimize

$$\mathsf{D}=\frac{1}{T}\int_0^T\mathbb{E}\{|\hat{S}(t)-S(t)|^2\}dt=\frac{1}{2}\mathbb{E}\|\mathbf{X}-\hat{\mathbf{X}}\|^2. \quad (50)$$

Since all the random variables are Gaussian, the linear MMSE is optimal and thus we would like to use $QL\mathbf{X}+\mathbf{Z}$ to find $\hat{\mathbf{X}}$ such that $E\|\mathbf{X}-\hat{\mathbf{X}}\|^2$ is minimized.

The mean square error is given by

$$\mathbb{E}\|\mathbf{X}-\hat{\mathbf{X}}\|^2=\mathbb{E}_{\mathbf{Y}}\{\mathsf{Var}[\mathbf{X}|\mathbf{Y}]\}=\mathtt{Tr}(C_e), \quad (51)$$

where the error covariance matrix C_e is of the form

$$\begin{aligned}
C_e&=C_{\mathbf{X}}-C_{\mathbf{XY}}C_{\mathbf{Y}}^{-1}C_{\mathbf{YX}} && (52)\\
&=C_{\mathbf{X}}-C_{\mathbf{X}}L^\dagger Q^\dagger(QLC_{\mathbf{X}}L^\dagger Q^\dagger+C_{\mathbf{Z}})^{-1}QLC_{\mathbf{X}}\\
&=\mathsf{p}I-\mathsf{p}^2L^\dagger Q^\dagger(\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2I)^{-1}QL. && (53)
\end{aligned}$$

In the above formulas, we have used the fact that $C_{\mathbf{X}}=\mathsf{p}I_{2N\times 2N}$ and $C_{\mathbf{Z}}=\sigma^2I_{M\times M}$. Therefore, from (50) we have

$$\mathsf{D}=\mathtt{Tr}(C_e)/2$$

and if we use (53), it follows that:

$$\begin{aligned}
2\mathsf{D}&=\mathtt{Tr}(C_e)\\
&=\mathtt{Tr}\left(\mathsf{p}I-\mathsf{p}^2L^\dagger Q^\dagger(\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2I)^{-1}QL\right) && (54)\\
&=2N\mathsf{p}-\mathsf{p}\cdot\mathtt{Tr}\left(\mathsf{p}L^\dagger Q^\dagger(\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2I)^{-1}QL\right)\\
&=2N\mathsf{p}-\mathsf{p}\cdot\mathtt{Tr}\left((\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2I)^{-1}(\mathsf{p}QLL^\dagger Q^\dagger)\right) && (55)\\
&=2N\mathsf{p}-\mathsf{p}\cdot\mathtt{Tr}\big((\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2I)^{-1}\\
&\qquad\times(\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2I-\sigma^2I)\big)\\
&=2N\mathsf{p}-\mathsf{p}\cdot\mathtt{Tr}\left(I-\sigma^2(\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2I)^{-1}\right) && (56)\\
&=(2N-M)\mathsf{p}+\mathsf{p}\sigma^2\mathtt{Tr}\left((\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2I)^{-1}\right), && (57)
\end{aligned}$$

where (55) results from the fact that the trace operator is invariant under cyclic permutations.

There is an alternative form of the Linear MMSE estimator ([21, p. 90]) in which the error covariance matrix C_e is of the form

$$C_e = (L^\dagger Q^\dagger C_{\mathbf{Z}}^{-1} QL + C_{\mathbf{X}}^{-1})^{-1} \tag{58}$$
$$= \mathsf{p}\sigma^2(\mathsf{p}L^\dagger Q^\dagger QL + \sigma^2 I)^{-1}. \tag{59}$$

Equivalently, D can be found using (59) as

$$2\mathsf{D} = \mathtt{Tr}(C_e) = \mathsf{p}\sigma^2\mathtt{Tr}(\mathsf{p}L^\dagger Q^\dagger QL + \sigma^2 I)^{-1}. \tag{60}$$

When $L = I$: We can express the two distortion formulas given in (57) and (60) in terms of the following two matrices:

$$\Delta_1 = \mathsf{p}QQ^\dagger + \sigma^2 I,$$
$$\Delta_2 = \mathsf{p}Q^\dagger Q + \sigma^2 I$$

as follows:

$$2\mathsf{D} = (2N - M)\mathsf{p} + \mathsf{p}\sigma^2\mathtt{Tr}\left(\Delta_1^{-1}\right) \tag{61}$$
$$= \mathsf{p}\sigma^2\mathtt{Tr}(\Delta_2^{-1}). \tag{62}$$

A direct calculation shows that the entries of matrix Δ_1 (an $M \times M$ matrix) are:

$$\Delta_{1(i,k)} = \mathsf{p}\left(\sum_{\ell=N_1}^{N_2}\cos(\ell\omega_0(t_i - t_k))\right) + \sigma^2\mathbf{1}[i = k], \tag{63}$$

where $\mathbf{1}[i = k]$ is one if $i = k$ and zero otherwise. Moreover, a direct calculation shows that the diagonal entries of matrix Δ_2 (a $2N \times 2N$ matrix) are given by:

$$\Delta_{2(k,k)} = \begin{cases} \sigma^2 + \mathsf{p}\sum_{i=1}^{M}\cos^2(\ell\omega_0 t_i); \\ \quad \ell = k + N_1 - 1 \text{ for } k = 1, \cdots, N, \\ \sigma^2 + \mathsf{p}\sum_{i=1}^{M}\sin^2(\ell\omega_0 t_i); \\ \quad \ell = k + N_1 - 1 - N \text{ for } k = N+1, \cdots, 2N. \end{cases} \tag{64}$$

C. Computing Variance of Distortion

The variance of distortion, using the Parseval's theorem given in (29), is of the form

$$\mathsf{V} = \mathsf{Var}\left\{\frac{1}{T}\int_0^T |\hat{S}(t) - S(t)|^2 dt\right\} = \frac{1}{2}\mathsf{Var}\{\|\mathbf{X} - \hat{\mathbf{X}}\|^2\}. \tag{65}$$

Using Theorem 7 from Appendix A, we have

$$\mathsf{Var}(\|\mathbf{X} - \hat{\mathbf{X}}\|^2) = 2\mathtt{Tr}(C_e^2). \tag{66}$$

Hence, the variance of distortion will be

$$\mathsf{V} = \mathtt{Tr}(C_e^2).$$

Using similar steps as the ones used in computing average distortion, from (53), V is found to be

$$\mathsf{V} = \frac{1}{2}\mathsf{Var}(\|\mathbf{X} - \hat{\mathbf{X}}\|^2) = \mathtt{Tr}(C_e^2)$$
$$= \mathtt{Tr}[\left(\mathsf{p}I - \mathsf{p}^2 L^\dagger Q^\dagger(\mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I)^{-1}QL\right)^2] \tag{67}$$
$$= \mathsf{p}^2[2N - M + \sigma^4\mathtt{Tr}((\mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I)^{-2})], \tag{68}$$

where the above equality holds because by defining $\Pi_{M\times 2N} = QL$, we can use the identity

$$\mathtt{Tr}[\left(\mathsf{p}I - \mathsf{p}^2\Pi^\dagger(\mathsf{p}\Pi\Pi^\dagger + \sigma^2 I)^{-1}\Pi\right)^2]$$
$$= \mathsf{p}^2[2N - M + \sigma^4\mathtt{Tr}((\mathsf{p}\Pi\Pi^\dagger + \sigma^2 I)^{-2})].$$

Alternatively, using the second form of MMSE given in (59), we get

$$\mathsf{V} = \mathtt{Tr}(C_e^2) = \mathsf{p}^2\sigma^4\mathtt{Tr}(\mathsf{p}L^\dagger Q^\dagger QL + \sigma^2 I)^{-2}. \tag{69}$$

When $L = I$: We can express the variance formulas in terms of matrices

$$\Delta_1 = \mathsf{p}QQ^\dagger + \sigma^2 I, \qquad \Delta_2 = \mathsf{p}Q^\dagger Q + \sigma^2 I$$

as follows:

$$\mathsf{V} = \mathsf{p}^2[2N - M + \sigma^4\mathtt{Tr}(\Delta_1^{-2})] \tag{70}$$
$$= \mathsf{p}^2\sigma^4\mathtt{Tr}(\Delta_2^{-2}). \tag{71}$$

So far, two closed formulas for D and V have been derived in equations (61), (62), (70) and (71). In the following subsections, the proof of the main results will be developed using the notation and formulas given in this subsection.

D. Proof of Theorem 1 (Section III)

We would like to show that setting $L = I$ is optimal. The coordinates of matrix L are determined by the LTI filter as given in equation (28). Observe that

$$LL^\dagger = \begin{pmatrix} L_1 & L_2 \\ -L_2 & L_1 \end{pmatrix}\begin{pmatrix} L_1 & -L_2 \\ L_2 & L_1 \end{pmatrix}$$
$$= \begin{pmatrix} {L_1}^2 + {L_2}^2 & 0 \\ 0 & {L_1}^2 + {L_2}^2 \end{pmatrix}, \tag{72}$$

where the matrix $L_1^2 + L_2^2$ is given on on top of the next page. It has diagonal entries that are less than or equal to one, by the passive filter condition. Therefore, from (72) and the assumption that the filter is passive, we obtain the constraint $LL^\dagger \leq I$, where by $A \leq B$ we mean that $B - A$ is positive semi-definite.

Hence, $Q(LL^\dagger - I)Q^\dagger \leq 0$, implying that

$$\mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I \leq \mathsf{p}QQ^\dagger + \sigma^2 I. \tag{73}$$

We now use the following inequality:

Theorem 8 (Klein's Inequality): [8] For any two symmetric positive definite matrices A and B and any differentiable convex function on $(0, \infty)$*, we have*

$$\mathtt{Tr}(f(A) - f(B)) \geq \mathtt{Tr}((A - B)f'(B)).$$

Let $f(x) = x^{-1}$ and $A = \mathsf{p}QQ^\dagger + \sigma^2 I$ and $B = \mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I$. Then from (73), we have $B \geq A$. Now, observe that $(A - B)f'(B) = (B - A)B^{-2}$ is the product of two positive semi-definite matrices. Hence, its trace is non-negative.[6] Thus, $\mathtt{Tr}[f(A) - f(B)] \geq 0$. We get that

$$\mathtt{Tr}(\mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I)^{-1} \geq \mathtt{Tr}(\mathsf{p}QQ^\dagger + \sigma^2 I)^{-1}. \tag{74}$$

[6]Observe that if A and B are positive semi-definite, then $\mathtt{Tr}(AB) = \mathtt{Tr}(AB^{1/2}B^{1/2}) = \mathtt{Tr}(B^{1/2}AB^{1/2}) \geq 0$.

$$L_1^2+L_2^2=\begin{pmatrix} H_R^2(N_1\omega_0)+H_I^2(N_1\omega_0) & 0 & \cdots & 0\\ 0 & H_R^2((N_1+1)+H_I^2((N_1+1)\omega_0) & 0 & 0\\ & \vdots & & \\ 0 & \cdots & 0 & H_R^2(N_2\omega_0)+H_I^2(N_2\omega_0)\end{pmatrix}$$

Not using any filter on the signal bandwidth is equivalent with $LL^\dagger = I$. Hence, pre-filtering is not helpful in reducing the MMSE. The pre-filtering is also not helpful in reducing variance of the distortion. It is shown that

$$\mathsf{Var}(\|\mathbf{X}-\hat{\mathbf{X}}\|^2) = 2\mathsf{p}^2(2N-M) + 2\mathsf{p}^2\sigma^4\mathtt{Tr}\left((\mathsf{p}QLL^\dagger Q^\dagger+\sigma^2 I)^{-2}\right),$$

and one can employ Klein's inequality for $f(x)=x^{-2}$ in the same manner.

E. Proof of Lemma 1 (Section III-A)

Since the noise variance $\sigma^2>0$, the matrix $\Delta_1=\mathsf{p}QQ^\dagger+\sigma^2 I$ is positive definite and real. In addition, from (63), all the diagonal entries of Δ_1 are $N\mathsf{p}+\sigma^2$. Applying Theorem 5 for the matrix Δ_1 and the convex function $f(x)=x^{-1}$ on $x>0$, we have

$$\mathtt{Tr}\left(\Delta_1^{-1}\right) \geq \sum_{j=1}^{M}\Delta_{1(j,j)}^{-1} = \frac{M}{N\mathsf{p}+\sigma^2}. \tag{75}$$

Substituting (75) into (61), results in a lower bound on distortion as follows

$$\begin{aligned} 2\mathsf{D} &= (2N-M)\mathsf{p}+\mathsf{p}\sigma^2\mathtt{Tr}\left(\Delta_1^{-1}\right)\\ &\geq (2N-M)\mathsf{p}+\mathsf{p}\sigma^2\frac{M}{N\mathsf{p}+\sigma^2}\\ &= (2N-M)\mathsf{p}+\mathsf{p}\frac{M}{1+SNR}.\end{aligned}$$

Note that we have defined $SNR = N\mathsf{p}/\sigma^2$. Because sampling time instances t_i are arbitrary, for any value of M, we obtain

$$\frac{\mathsf{D}_{\min}}{\mathsf{p}} \geq \frac{1}{2}\left(2N-M+\frac{M}{1+SNR}\right). \tag{76}$$

Since $f(x)=x^{-1}$ is a strictly convex function for $x>0$, equality in the above equation holds if and only if $\Delta_1 = \mathsf{p}QQ^\dagger+\sigma^2 I$ is a diagonal matrix. Therefore, using (63), we have the following equation for $1\leq k<i\leq M$ (remember that $\omega_0=2\pi/T$):

$$\begin{aligned} \Delta_{1(i,k)} &= \sum_{\ell=N_1}^{N_2}\cos(\ell\omega_0(t_i-t_k)) = \mathsf{Real}\{\sum_{\ell=N_1}^{N_2}e^{j2\pi\ell\frac{t_i-t_k}{T}}\}\\ &= \mathsf{Real}\{e^{j\pi\frac{t_i-t_k}{T}(N_1+N_2)}\ \frac{\sin(\pi\frac{t_i-t_k}{T}N)}{\sin(\pi\frac{t_i-t_k}{T})}\}\\ &= \cos(\pi\frac{t_i-t_k}{T}(N_1+N_2))\ \frac{\sin(\pi\frac{t_i-t_k}{T}N)}{\sin(\pi\frac{t_i-t_k}{T})} = 0.\end{aligned}$$

Consequently, the sampling time instances t_i should satisfy the following equations:

$$\begin{cases}\sin(\pi N\frac{(t_i-t_k)}{T})=0,\\ \quad\text{or}\\ \cos(\pi(N_1+N_2)\frac{(t_i-t_k)}{T})=0.\end{cases} \tag{77}$$

Thus, given any i and j, we should either have

$$|t_i-t_j| = T\frac{m_1}{N},\quad \text{for some integer } m_1,$$

or

$$|t_i-t_j| = T\frac{2m_2+1}{2(N_1+N_2)},\quad \text{for some integer } m_2.$$

F. Derivation of the Lower Bound for Variance

Similarly, the lower bound on the variance is developed using (70) and Theorem 5 for the convex function $f(x)=x^{-2}$ on $x>0$. From Theorem 5, we have

$$\mathtt{Tr}\left(\Delta_1^{-2}\right) \geq \sum_{j=1}^{M}\Delta_{1(j,j)}^{-2} = \frac{M}{(N\mathsf{p}+\sigma^2)^2}. \tag{78}$$

And thus using (70), we can conclude that

$$\begin{aligned}\mathsf{V}_{\min} &\geq \mathsf{p}^2\left[2N-M+\sigma^4\frac{M}{(N\mathsf{p}+\sigma^2)^2}\right]\\ &= \mathsf{p}^2\left[2N-M+\frac{M}{(1+SNR)^2}\right].\end{aligned} \tag{79}$$

Here again, equality holds if and only if $\Delta_1=\mathsf{p}QQ^\dagger+\sigma^2 I$ is a diagonal matrix.

G. Proof of Theorem 2 (Section III-A)

Here we are in the case of $M\leq N$. In Lemma 1, the desired lower bounds on $\mathsf{D}_{\min}$ and $\mathsf{V}_{\min}$ were developed. Moreover, we showed that these lower bounds are achievable if for any time instances t_i or t_j

$$|t_i-t_j| = T\frac{m_1}{N},\quad \text{for some integer } m_1, \tag{80}$$

$$\textit{or}\quad |t_i-t_j| = T\frac{2m_2+1}{2(N_1+N_2)},\quad \text{for some integer } m_2. \tag{81}$$

From (80), we conclude that any arbitrary choice of time instances from the set

$$\{\tau,\tau+\frac{1}{N}T,\tau+\frac{2}{N}T,\tau+\frac{3}{N}T,\cdots,\tau+\frac{N-1}{N}T\}$$

is optimal.

To find the interpolation formula, consider the MMSE estimator of $\mathbf{X}$ from $\mathbf{Y}=Q\mathbf{X}+\mathbf{Z}$. The estimator $\hat{\mathbf{X}}$ that minimizes $E\|\mathbf{X}-\hat{\mathbf{X}}\|^2$ can be written as $W\mathbf{Y}$, where $W = C_{\mathbf{XY}}C_{\mathbf{Y}}^{-1} = \mathsf{p}Q^\dagger\Delta_1^{-1}$. Since with the optimal choice of t_i, $\Delta_1 = (N\mathsf{p}+\sigma^2)I$, the Fourier coefficients of the reconstructed signal are as follows:

$$\hat{\mathbf{X}} = \frac{\mathsf{p}}{N\mathsf{p}+\sigma^2}Q^\dagger\mathbf{Y}. \tag{82}$$

The above formula for $\hat{\mathbf{X}}$ results in the desired reconstruction formula in (13).

H. Proof of Proposition 1 (Section III-A)

Remember the lower bounds given in Lemma 1. We showed that equality in this lemma holds if and only if

$$|t_i - t_j| = T\frac{m}{N}, \quad \text{for some integer } m, \tag{83}$$

or

$$|t_i - t_j| = T\frac{2m+1}{2(N_1+N_2)}, \quad \text{for some integer } m. \tag{84}$$

It remains to show that when $M \le N$ and N does not divide N_1+N_2, we have the following statement: one can find $M-1$ sampling times such that their pairwise differences satisfy (83). Without loss of generality assume that $t_1 = 0$. For any t_i, t_j, the differences $t_i - t_1$, $t_j - t_1$ cannot simultaneously satisfy (84), since then $t_i - t_j$ will be of the form $T(2m_i - 2m_j)/(2(N_1+N_2))$ and this pairwise distance does not satisfy either of (83) or (84). Therefore, at least $M-1$ points should satisfy $t_i - t_1$ with (83). This completes the proof.

I. Proof of Lemma 2 (Section III-B)

We use the distortion formula given in (62) and minimize the distortion subject to the sampling locations. We have

$$\begin{aligned}
\mathsf{D}_{\min} &= \min_{\{t_i, i=1,\cdots,M\}} \frac{1}{2}\mathsf{p}\sigma^2\mathtt{Tr}(\Delta_2^{-1}) \\
&\ge \frac{1}{2}\mathsf{p}\sigma^2 \min_{\{t_i, i=1,\cdots,M\}} \sum_{k=1}^{2N}\Delta_{2(k,k)}^{-1} && (85)\\
&= \frac{1}{2}\mathsf{p}\sigma^2 \min_{\{t_i, i=1,\cdots,M\}} \sum_{\ell=N_1}^{N_2}\frac{1}{\eta_\ell}+\frac{1}{\gamma_\ell}, && (86)\\
&\ge \frac{1}{2}\mathsf{p}\sigma^2 \min_{\{t_i, i=1,\cdots,M\}} \sum_{\ell=N_1}^{N_2}\frac{2}{(\eta_\ell+\gamma_\ell)/2} && (87)\\
&= \frac{1}{2}\mathsf{p}\sigma^2 \sum_{\ell=N_1}^{N_2}\frac{2}{\sigma^2+\frac{M\mathsf{p}}{2}} \\
&= \frac{N\sigma^2\mathsf{p}}{\sigma^2+\frac{M\mathsf{p}}{2}} = \frac{N\mathsf{p}}{1+\frac{M}{2N}SNR}, && (88)
\end{aligned}$$

where (85) follows from Theorem 5 for the convex function $f(x) = x^{-1}$ for $x > 0$; (86) is written using (64) where η_ℓ and γ_ℓ are defined as

$$\begin{aligned}
\eta_\ell &= \sigma^2 + \mathsf{p}\sum_{i=1}^{M}\cos^2(\ell\omega_0 t_i), \\
\gamma_\ell &= \sigma^2 + \mathsf{p}\sum_{i=1}^{M}\sin^2(\ell\omega_0 t_i). && (89)
\end{aligned}$$

Furthermore, (87) follows from the inequality $a^{-1}+b^{-1} \ge 4(a+b)^{-1}$ for non-negative a and b.

The proof for the variance is similar. Using (69), and Theorem 5 for the convex function $f(x) = x^{-2}$ for $x > 0$, we have

$$\begin{aligned}
\mathsf{V}_{\min} &= \min_{\{t_i, i=1,\cdots,M\}} [\mathsf{p}^2\sigma^4\mathtt{Tr}(\Delta_2^{-2})] \\
&\ge \mathsf{p}^2\sigma^4 \min_{\{t_i, i=1,\cdots,M\}} \sum_{k=1}^{2N}\Delta_{2(k,k)}^{-2} \\
&= \mathsf{p}^2\sigma^4 \min_{\{t_i, i=1,\cdots,M\}} \sum_{\ell=N_1}^{N_2}\frac{1}{\eta_\ell^2}+\frac{1}{\gamma_\ell^2} && (90)\\
&\ge \mathsf{p}^2\sigma^4 \min_{H(.),\{t_i, i=1,\cdots,M\}} \sum_{\ell=N_1}^{N_2}\frac{8}{(\eta_\ell+\gamma_\ell)^2} \\
&= \mathsf{p}^2\sigma^4 \sum_{\ell=N_1}^{N_2}\frac{8}{(2\sigma^2+M\mathsf{p})^2} \\
&= N\mathsf{p}^2\sigma^4\frac{8}{(2\sigma^2+M\mathsf{p})^2} \\
&= N\mathsf{p}^2\frac{2}{(1+\frac{M\mathsf{p}}{2\sigma^2})^2} \\
&= \frac{2N\mathsf{p}^2}{(1+\frac{M}{2N}SNR)^2} && (91)
\end{aligned}$$

where η_ℓ and γ_ℓ are given in (89), and (91) follows from $a^{-2}+b^{-2} \ge 8(a+b)^{-2}$, for nonzero values of a and b.

Necessary and Sufficient Conditions for Tightness of the Lower Bounds: For Theorem 5 to be tight, we need $\Delta_2 = \mathsf{p}Q^\dagger Q + \sigma^2 I$ to be a diagonal matrix. We further need $\eta_\ell = \gamma_\ell$ to have all the inequalities as equalities. Therefore, the equality holds if and only if matrix Δ_2 is diagonal, and all the diagonal entries are equal.

The off-diagonal and diagonal entries of Δ_2 are respectively given in (92) and (93) on top of the next page.

If we put the off-diagonal entries zero and write the above equations in a simpler form, we get

$$\begin{cases}
\sum_{\ell=1}^{M}\cos((N_1+k_1)\omega_0 t_\ell)\cos((N_1+k_2)\omega_0 t_\ell) = 0; \\
\qquad\qquad 0 \le k_1 < k_2 \le N-1 \\
\sum_{\ell=1}^{M}\cos((N_1+k_1)\omega_0 t_\ell)\sin((N_1+k_2)\omega_0 t_\ell) = 0; \\
\qquad\qquad 0 \le k_1, k_2 \le N-1 \\
\sum_{\ell=1}^{M}\sin((N_1+k_1)\omega_0 t_\ell)\sin((N_1+k_2)\omega_0 t_\ell) = 0; \\
\qquad\qquad 0 \le k_1 < k_2 \le N-1.
\end{cases} \tag{94}$$

Substituting $\omega_0 = 2\pi/T$ in the above equations, we obtain

$$\begin{cases}
\sum_{\ell=1}^{M} e^{j2\pi(k_1-k_2)\frac{t_\ell}{T}} = 0; & 0 \le k_1 < k_2 \le N-1 \\
\sum_{\ell=1}^{M} e^{j2\pi(2N_1+k_1+k_2)\frac{t_\ell}{T}} = 0; & 0 \le k_1 \le k_2 \le N-1,
\end{cases}$$

or in another form, we have

$$\begin{cases}
\sum_{i=1}^{M} e^{j2\pi k\frac{t_i}{T}} = 0 & \text{for} 0 < k \le N-1 \\
\sum_{i=1}^{M} e^{j2\pi k\frac{t_i}{T}} = 0 & \text{for} 2N_1 \le k \le 2N_2.
\end{cases} \tag{95}$$

$$\Delta_{2(i,k)} = \begin{cases} \mathsf{p}\cdot\sum_{\ell=1}^{M}\cos((N_1+i-1)\omega_0 t_\ell)\cos((N_1+k-1)\omega_0 t_\ell); & 1\le i<k\le N \\ \mathsf{p}\cdot\sum_{\ell=1}^{M}\cos((N_1+i-1)\omega_0 t_\ell)\sin((N_1+k-N)\omega_0 t_\ell); & 1\le i\le N\le k\le 2N \\ \mathsf{p}\cdot\sum_{\ell=1}^{M}\sin((N_1+i-N)\omega_0 t_\ell)\sin((N_1+k-N)\omega_0 t_\ell); & N\le i\le 2N,\ N\le k\le 2N, \end{cases} \tag{92}$$

$$\Delta_2(i,i) = \begin{cases} \frac{M\mathsf{p}}{2}+\sigma^2+\frac{\mathsf{p}}{2}\cdot\sum_{\ell=1}^{M}\cos(2(N_1+i-1)\omega_0 t_\ell); & 1\le i\le N \\ \frac{M\mathsf{p}}{2}+\sigma^2-\frac{\mathsf{p}}{2}\cdot\sum_{\ell=1}^{M}\cos(2(N_1+i-1)\omega_0 t_\ell); & N\le i\le 2N, \end{cases} \tag{93}$$

These equations imply that the off-diagonal entries are all zero, and specifically,

$$\sum_{\ell=1}^{M}\cos(2(N_1+i-1)\omega_0 t_\ell)=0, \qquad 1\le i\le 2N.$$

Thus, from (93), the diagonal entries $\Delta_{2(i,i)}$ are all equal to $M\mathsf{p}/2+\sigma^2$.

J. Proof of Theorem 3 (Section III-B)

Consider the proof of Lemma 2 in which the alternative lower bound on the average and variance of distortion, and necessary and sufficient conditions for their tightness were derived. With uniform sampling, *i.e.,* $t_i = \frac{iT}{M}, i = 1, 2, \cdots, M$, when $2N_1+\alpha+\beta$ does not divide M for integers $0\le\alpha,\beta\le N-1$, matrix Δ_2 will be diagonal with diagonal entries $(\sigma^2+\frac{M\mathsf{p}}{2})$. Thus this lower bounds will be tight.
To find the interpolation formula, consider the MMSE estimator of $\mathbf{X}$ from $\mathbf{Y} = Q\mathbf{X}+\mathbf{Z}$. The estimator $\hat{\mathbf{X}}$ that minimizes $E\|\mathbf{X}-\hat{\mathbf{X}}\|^2$ can be written as $W\mathbf{Y}$, where $W = C_{\mathbf{XY}}C_{\mathbf{Y}}^{-1} = \mathsf{p}\Delta_2^{-1}Q^\dagger$. For the optimal sampling locations t_i, $\Delta_2 = (Mp/2+\sigma^2)I$, and hence

$$\hat{\mathbf{X}} = W\mathbf{Y} = \frac{\mathsf{p}}{\frac{M\mathsf{p}}{2}+\sigma^2}Q^\dagger\mathbf{Y}. \tag{96}$$

The estimated coefficient vector, $\hat{\mathbf{X}}$, results in the reconstruction formula given in (19).

K. Proof of Theorem 4 (Section III-C)

Here we are in the case of $N < M \le 2N$. To prove the upper bound, we choose not to utilize a pre-sampling filter and use the following M sampling points

$$\{0, \frac{1}{N}T, \frac{2}{N}T, \frac{3}{N}T, \cdots, \frac{N-1}{N}T, \times\frac{1}{2N}T, \frac{3}{2N}T, \cdots, \frac{2M-2N-1}{2N}T\}. \tag{97}$$

Using (63), one can verify that

$$\Delta_1 = \begin{bmatrix} (N\mathsf{p}+\sigma^2)\ \mathbf{I}_{N\times N} & \mathsf{p}\ \mathbf{G}_{N\times(M-N)} \\ \mathsf{p}\ \mathbf{G}^\dagger_{(M-N)\times N} & (N\mathsf{p}+\sigma^2)\ \mathbf{I}_{(M-N)} \end{bmatrix}_{M\times M},$$

where $\mathbf{G}$ is an $N\times(M-N)$ matrix whose $(M-N)$ columns are the first $(M-N)$ columns of the matrix $\Phi_{N\times N}$ defined as follows: Φ is an $N\times N$ circulant matrix with the first row $\mathbf{c} = [c_0,\ c_1,\ \ldots,\ c_{N-1}]$, where

$$c_k = \frac{1}{2}\sum_{\ell=N_1}^{N_2}\left(\omega^{-(2k+1)\ell}+\omega^{(2k+1)\ell}\right) \tag{98}$$

in which $\omega = \exp(j\pi/N)$. For instance, when $N_1 = 1$, Φ is an $N\times N$ matrix with all -1 entries.

Using Theorem 9 from Appendix B-H1, the eigenvalues of $\Delta_1-(N\mathsf{p}+\sigma^2)\mathbf{I}$ are equal to ps_i and $-ps_i$ (where s_i are the singular values of $\mathbf{G}_{N\times(M-N)}$), in addition to $M-2(M-N) = 2N-M$ zero eigenvalues. One can find the eigenvalues of Δ_1 by adding $N\mathsf{p}+\sigma^2$ to the eigenvalues of $\Delta_1-(N\mathsf{p}+\sigma^2)\mathbf{I}$, and from that we have

$$\begin{aligned}\mathtt{Tr}(\Delta_1^{-1}) &= \frac{2N-M}{N\mathsf{p}+\sigma^2} + \sum_{i=1}^{M-N}\left(\frac{1}{N\mathsf{p}+\sigma^2-ps_i}+\frac{1}{N\mathsf{p}+\sigma^2+ps_i}\right) \\ &= \frac{2N-M}{N\mathsf{p}+\sigma^2}+\sum_{i=1}^{M-N}\frac{2(N\mathsf{p}+\sigma^2)}{(N\mathsf{p}+\sigma^2)^2-p^2s_i^2}.\end{aligned} \tag{99}$$

To compute an upper bound on $\mathtt{Tr}(\Delta_1^{-1})$, we need to find an upper bound on s_i. To do so, first we assume that $M = 2N$, and find the singular values of the circulant matrix $\Phi_{N\times N}$ (named δ_i for $i = 1,\cdots,N$) and then use the upper bound given in Theorem 12 on the singular values of the matrix $\mathbf{G}_{N\times(M-N)}$.

Singular values of Φ are the eigenvalues of $(\Phi\Phi^\dagger)^{1/2}$. Theorem 10 from Appendix B-H1 states that any two circulant matrices commute and the eigenvalues of their product is the pairwise product of their eigenvalues. Since both Φ and $\Phi^\dagger$ are circulant matrices, we conclude that $\Phi\Phi^\dagger$ is also a circulant matrix with eigenvalues $|\lambda_i|^2, i = 1,\ldots,N$, where λ_i are eigenvalues of Φ. Thus, $\delta_i = |\lambda_i|$ for $i = 1,\cdots,N$. Using Theorem 11 from Appendix B-H1, λ_i are given by

$$\lambda_i = \sum_{k=0}^{N-1}c_k\omega^{-2ki}, \qquad i = 1, 2, \cdots, N \tag{100}$$

where $\omega = \exp(j\pi/N)$. Substituting c_k from (98) into (100), we obtain

$$\begin{aligned}\lambda_i &= \frac{1}{2}\sum_{k=0}^{N-1}\ \Big[\sum_{\ell=N_1}^{N_2}\left(\omega^{-(2k+1)\ell}+\omega^{(2k+1)\ell}\right)\Big]\ \omega^{-2ki} \\ &= \frac{1}{2}\sum_{\ell=N_1}^{N_2}\omega^{-\ell}\sum_{k=0}^{N-1}\omega^{-2k(\ell+i)}+\frac{1}{2}\sum_{\ell=N_1}^{N_2}\omega^{\ell}\sum_{k=0}^{N-1}\omega^{2k(\ell-i)}.\end{aligned}$$

We then have

$$\sum_{k=0}^{N-1}\omega^{-2k(\ell+i)} = \begin{cases} N; & N\mid\ell+i \\ 0; & N\nmid\ell+i, \end{cases} \tag{101}$$

and

$$\sum_{k=0}^{N-1} \omega^{2k(\ell-i)} = \begin{cases} N; & N|\ell - i \\ 0; & N \nmid \ell - i. \end{cases} \tag{102}$$

Given any arbitrary i, $\ell + i \in [N_1 + i : N_2 + i]$ are N consecutive numbers, and only one of them is divisible by N. Let ℓ_1, ℓ_2 be unique numbers in $[N_1 : N_2]$ such that $N|\ell_1 + i$ and $N|\ell_2 - i$. We then have

$$\begin{aligned}\lambda_i &= \frac{N}{2}\omega^{-\ell_1} + \frac{N}{2}\omega^{\ell_2} \\ &= \frac{N}{2}\omega^{(\ell_2-\ell_1)/2}(\omega^{-(\ell_2+\ell_1)/2} + \omega^{(\ell_2+\ell_1)/2}) \\ &= N\omega^{(\ell_2-\ell_1)/2}\cos(\frac{(\ell_2+\ell_1)\pi}{2N}).\end{aligned}$$

Thus, $|\lambda_i| = N|\cos(\frac{(\ell_2+\ell_1)\pi}{2N})|$. From $N|\ell_1 + i$ and $N|\ell_2 - i$, we have that $N|\ell_1 + \ell_2$. But $\ell_1 + \ell_2 \in [2N_1 : 2N_2]$ consists of $2N_2 - 2N_1 + 1 = 2N - 1$ consecutive natural numbers, and there cannot be more than two numbers that are divisible by N in $[2N_1 : 2N_2]$. If $\ell_1 + \ell_2 = kN$, we will have

$$|\lambda_i| = N|\cos(\frac{k\pi}{2})| = \begin{cases} N; & k \text{ is even} \\ 0; & k \text{ is odd.} \end{cases}$$

Therefore, we need to find out for the number of values of i, $|\lambda_i|$ is zero, and for the number values of i that $|\lambda_i|$ is N.

Assume that $N_1 = qN + r$. Then $N_1 + i = qN + r + i$ and $N_1 - i = qN + r - i$. Therefore

$$\ell_1 = \begin{cases} N_1 + N - (r+i) & r+i \leq N \\ N_1 + 2N - (r+i) & r+i > N, \end{cases} \tag{103}$$

$$\ell_2 = \begin{cases} N_1 & i-r = N \\ N_1 + i - r & N-1 \geq i-r \geq 0 \\ N_1 + N - (r-i) & i-r < 0. \end{cases} \tag{104}$$

By considering different cases, one gets that the number of values of i where $|\lambda_i|$ is N is equal to $f(N_1, N)$, where

$$f(a,b) = \begin{cases} b-1 & \text{if } r = 0 \\ 2b - 2r + 1 & \text{if } 2r > b \\ 2r - 1 & \text{if } 0 < 2r \leq b, \end{cases}$$

where q and r are the quotient and remainder of dividing a by b.

Now for $N < M \leq 2N$, we use Theorem 12 from Appendix B-H1 for matrix Φ with singular values $\delta_1 \geq \delta_2, \cdots \geq \delta_N$ and submatrix $\mathbf{G}_{N\times(M-N)}$ with singular values $s_1 \geq s_2, \cdots \geq s_{M-N}$. Therefore, $\delta_i \geq s_i$ for $i = 1, 2, \cdots, M-N$, thus, $\mathbf{G}_{N\times(M-N)}$ has at most $\mathsf{Num} = \min\left(f(N_1, N), M-N\right)$ non-zero singular values. Furthermore the absolute value of non-zero eigenvalues of $\mathbf{G}$ is less than or equal to N. Using this upper bound on s_i and substituting it into (99), we get

$$\begin{aligned}\mathtt{Tr}(\Delta_1^{-1}) &= \frac{2N-M}{N\mathsf{p}+\sigma^2} + \sum_{i=1}^{M-N}\frac{2(N\mathsf{p}+\sigma^2)}{(N\mathsf{p}+\sigma^2)^2 - p^2 s_i^2} & (105)\\ &\leq \frac{2N-M}{N\mathsf{p}+\sigma^2} + \mathsf{Num}\frac{2(N\mathsf{p}+\sigma^2)}{(N\mathsf{p}+\sigma^2)^2 - p^2N^2} \\ &\quad + (M-N-\mathsf{Num})\frac{2}{(N\mathsf{p}+\sigma^2)}, & (106)\end{aligned}$$

Thus, (61) would be

$$\begin{aligned}2D &= (2N-M)\mathsf{p} + \mathsf{p}\sigma^2\cdot\mathtt{Tr}(\Delta_1^{-1}) & (107)\\ &\leq (2N-M)\mathsf{p} + \mathsf{p}\sigma^2 \\ &\quad\times\Big\{\frac{2N-M}{N\mathsf{p}+\sigma^2} + \mathsf{Num}\cdot\frac{2(N\mathsf{p}+\sigma^2)}{(N\mathsf{p}+\sigma^2)^2 - p^2N^2} \\ &\quad + (M-N-\mathsf{Num})\cdot\frac{2}{(N\mathsf{p}+\sigma^2)}\Big\}, & (108)\end{aligned}$$

which results in the following upper bound:

$$\frac{\mathsf{D}_{\min}}{\mathsf{p}} \leq \frac{1}{2}(2N-M) + \frac{2N-M}{2(1+SNR)} + \mathsf{Num}\cdot\frac{1+SNR}{1+2\ SNR} + (M-N-\mathsf{Num})\cdot\frac{1}{1+SNR}.$$

This concludes the proof of the upper bound.

1) Results Needed for the Proof of Theorem 4:

Theorem 9 [22 Ex. 6, 17-2]: Consider the Jordan–Wielandt matrix of the block form $P = \begin{bmatrix} \mathbf{0} & G_{m\times n} \\ G^\dagger & \mathbf{0}\end{bmatrix}$. *Assume that* $\{s_i(G)\}$ *are the singular values of* G. *Then, the eigenvalues of* P *are* $\{\pm s_i(G)\}$ *together with* $|m-n|$ *zeros.*

Theorem 10 (23 p. 34]: Every $n \times n$ *circulant matrix* $C = [c_{k-j}]$ *has eigenvectors* $y^{(m)} = \frac{1}{\sqrt{n}}[1, e^{-j2\pi m/n}, \ldots, e^{-j2\pi(n-1)m/n}]$, *for* $m = 0, 1, \cdots, n-1$ *and corresponding eigenvalues* $\psi_m = \sum_{k=0}^{n-1} c_k e^{-j2\pi mk/n}$, *and can be expressed in the form* $C = U\Psi U^*$, *where* U *has the eigenvectors as columns in order and* $\Psi = diag(\psi_m)$ *is a diagonal matrix with diagonal elements* $\psi_0, \psi_1, \cdots, \psi_n - 1$.

Theorem 11 (23 p. 35): Let $B = [b_{k-j}]$ *and* $C = [c_{k-j}]$ *be two* $n \times n$ *circulant matrices with eigenvalues* $\beta_m = \sum_{k=0}^{n-1} b_k e^{-j2\pi mk/n}$, $\psi_m = \sum_{k=0}^{n-1} c_k e^{-j2\pi mk/n}$, *respectively. Then matrices* B *and* C *commute and* $BC = CB = U\Psi U^*$, *where* $\Psi = diag(\beta_m\psi_m)$, *and* BC *is also a circulant matrix.*

Theorem 12 ([24]): Let A *be an* $m\times n$ *matrix with singular values* $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\min(m,n)}$. *Let* B *be a* $p\times q$ *submatrix of* A *(intersection of any* p *rows and any* q *columns of* A*) with singular values* $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{\min(p,q)}$. *Then,*

$$\begin{cases} \alpha_i \geq \beta_i & ; i = 1, 2, \cdots, \min(p,q) \\ \beta_i \geq \alpha_{i+(m-p)+(n-q)} & ; i \leq \min(p+q-m, p+q-n) \end{cases}.$$

L. Proof of Theorem 6 (Section V-A)

Since we have a pre-sampling filter $H(\omega)$, we use the average distortion formula given in (57):

$$\begin{aligned}2\mathsf{D} &= (2N-M)\mathsf{p} + \mathsf{p}\sigma^2 \mathtt{Tr}\left((\mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I)^{-1}\right)\\ &= M\mathsf{p} + \mathsf{p}\sigma^2 \mathtt{Tr}\left((\mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I)^{-1}\right)\end{aligned}$$

One can verify that the diagonal entries of the matrix $\mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I$ are $M\mathsf{p}/2 + \sigma^2$, but this matrix is not diagonal as

$$\begin{aligned}(\mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I)_{(2,1)} &= \mathsf{p}\sum_{\ell=N_1}^{N_2} a_\ell \cos(\ell\omega_0(t_2-t_1))\\ &= \mathsf{p}\sum_{\ell=N_1}^{N_1+M/2-1} \cos(\frac{\ell\omega_0 T}{M})\\ &= \mathsf{p}\sum_{\ell=N_1}^{N_1+M/2-1} \cos(\frac{2\pi\ell}{M})\\ &= -\mathsf{p}\frac{\sin\left(\frac{(2N_1-1)\pi}{2M}\right)}{\sin(\frac{\pi}{M})} \neq 0,\end{aligned}$$

since M is even. Hence the equality condition in the Theorem 5 is not satisfied and

$$\begin{aligned}\mathsf{D}_H &= \frac{1}{2}\mathsf{p}\left(M + \sigma^2 \mathtt{Tr}\left((\mathsf{p}QLL^\dagger Q^\dagger + \sigma^2 I)^{-1}\right)\right)\\ &> \frac{1}{2}\mathsf{p}\left(M + \sigma^2\frac{M}{\frac{M\mathsf{p}}{2}+\sigma^2}\right) = \frac{1}{2}\mathsf{p}\left(M + \frac{M}{1+\frac{1}{2}SNR}\right).\end{aligned}$$

M. Proof of Proposition 2 (Section V-B)

To study non-uniform power constraints, we can use the same mathematical framework given in Section V-A. The only change is that $C_{\mathbf{X}}$ is no longer equal to $\mathsf{p}I$. Here $C_{\mathbf{X}}$ is a diagonal matrix with the following diagonal entries

$$C_{\mathbf{X}(k,k)} = \begin{cases}\mathsf{p}_{k+N_1-1}; & k = 1,\cdots,N\\ \mathsf{p}_{k+N_1-1-N}; & k = N+1,\cdots,2N.\end{cases} \tag{109}$$

Therefore, (58) results in

$$\begin{aligned}C_e &= (Q^\dagger C_{\mathbf{Z}}^{-1} Q + C_{\mathbf{X}}^{-1})^{-1}\\ &= (\frac{1}{\sigma^2}Q^\dagger Q + C_{\mathbf{X}}^{-1})^{-1} = \mathsf{p}\sigma^2(\mathsf{p}Q^\dagger Q + \mathsf{p}\sigma^2 C_{\mathbf{X}}^{-1})^{-1}.\end{aligned} \tag{110}$$

Let $\Delta_2' = Q^\dagger Q + \sigma^2 C_{\mathbf{X}}^{-1}$. Observe that when $\mathsf{p}_\ell = \mathsf{p}$ for all ℓ, A_2' reduces to Δ_2. In fact, since $C_{\mathbf{X}}^{-1}$ is a diagonal matrix, the proof of Lemma 2 can be directly mimicked here, with no essential changes. Similarly, the proof of Theorem 3 can be adapted in a straightforward manner to show that uniform sampling, *i.e.,* $t_i = iT/M, i = 1,2,\cdots,M$ is a solution to the above equation if for each k in the interval $2N_1 \le k \le 2N_2$, M does not divide k. Particularly, for the rates above the Nyquist rate, *i.e.,* $M > 2N_2$, uniform sampling is optimal.

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Elaheh Mohammadi is currently a PhD student at Sharif University of Technology, Tehran, Iran. She obtained her Msc and Bsc degrees from Amirkabir and Tehran Universities in Electrical Engineering, respectively. Her research interests are sampling theory and signal processing.

Farokh Marvasti received his BS, Ms and PhD degrees all from Rensselaer Polytechnic Institute in 1970, 1971 and 1973, respectively. He has worked, consulted and taught in various industries and academic institutions since 1972. Among which are Bell Labs, University of California Davis, Illinois Institute of Technology, University of London, King's College. He was one of the editors and associate editors of IEEE Trans on Communications and Signal Processing from 1990-1997. He has about 100 Journal publications and has written several reference books; he has also several international patents. His last book is on *Non-uniform Sampling: Theory and Practice* by Springer in 2001. He was also a guest editor on Special Issue on *Non-uniform Sampling for the Sampling Theory and Signal and Image Processing Journal*, May 2008. Besides being the co-founders of two international conferences (ICT's and SampTA's), he has been the organizer and special session chairs of many IEEE conferences including ICASSP conferences. Recently, he was the Lead editor on "Sparse Signal Processing" for the Special Issue of *EURASIP Journal on Advanced in Signal Processing*. Dr. Marvasti is currently a professor at Sharif University of Technology and the director of Advanced Communications Research Institute (ACRI) and a former head of Center for Multi-Access Communications Systems. He spent his sabbatical leave at the Communications and Information Systems Group of University College London (UCL) in 2013. Prof Marvasti received a distinguishd award from the Iranian Academy of Sciences in 2014 and a 5 year term chair position from Iranian National Science foundation in 2015.