Statistical definition of locking bandwidth in an array of synchronised microwave oscillators

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Abstract: Arrays of N weakly coupled oscillators are considered in different configurations. The locking bandwidth for these arrays is defined statistically. Various factors affecting the locking bandwidth and the effect of the coupling network topology on locking bandwidth are studied by solving the dynamic equations of the array numerically. The analytical locking bandwidth for two configurations of arrays is computed and the results are compared with the numerical solution of dynamic equations.

1 Introduction

Synchronisation is an efficient method for power combining of many oscillators in microwaves and millimetrewaves [1]. Inter-injection locking is obtained by the establishment of bilateral coupling between the oscillators in the array [2, 3]. Obviously, the ideal case is one, in which all the oscillators have the same free-running frequency, however, in practice, it is difficult, if not impossible, to have exactly similar oscillators. On the other hand, if the difference between the free-running frequencies of the oscillators is greater than a certain value, synchronising would become impossible. For two mutually coupled oscillators, the locking bandwidth has been defined previously [4, 5]. Recently, a worst-case distribution of free-running frequencies of oscillators in arrays of coupled oscillators has been considered and it was shown that the locking bandwidth for circular arrangement of coupled oscillators is twice as that of their linear arrangement [6]. However, a more realistic assumption is that the free-running frequencies are random variables with a Gaussian (normal) distribution around an average value. In this paper, we extend the definition of locking bandwidth from the statistical point of view. The locking bandwidth is defined as the standard deviation of the free-running frequencies in which the probability of locking (PL) becomes greater than a certain value. Then the PL is obtained analytically for some types of coupled oscillator networks and the results are confirmed numerically by solving the related differential equations.

2 Mutually coupled oscillators

In an array of oscillators, each oscillator is coupled weakly to other oscillators. There are different arrangements of the coupling networks as depicted in Fig. 1. Each oscillator is represented by a simple shunt model (Fig. 2). $I_{\text{inj},n}$ is the injection current phasor in the oscillator n as a vector summation of all other oscillator inductions. Writing KCL in each oscillator's node and assuming the quadratic variation form for the nonlinear negative conductance, one obtains the following differential equation [7]

$$\frac{\mathrm{d}I_n}{\mathrm{d}t} = I_n\left[\frac{\mu\,\omega_n}{2Q}\left(\alpha_{n0}^2 - |I_n|^2\right) + \mathrm{j}\omega_n\right] + \frac{\omega_n}{2Q}\sum_{m=1}^{N}\kappa_{nm}I_m,\quad n = 1, 2, \ldots, N \tag{1}$$

where I_n is the current phasor of the load in the nth oscillator, ω_n the free-running oscillation frequency of the nth oscillator, Q the external quality factor of each oscillator (assumed to be equal for all oscillators), μ the saturation factor of nonlinear element, α_{n0} the free-running amplitude of oscillation of the nth oscillator and κ_{nm} the coupling coefficient between oscillators n and m.

Assuming $I_n = A_n\mathrm{e}^{\mathrm{j}\theta n}$, by separating the real and imaginary parts of (1), we arrive at two sets of differential equations for the amplitude and phase of the oscillations. Assuming a weak coupling between oscillators ($\kappa_{nm} \equiv \varepsilon_{nm}\mathrm{e}^{-\mathrm{j}\Phi_{\text{nm}}}$, $\varepsilon_{nm} \langle\langle 1$), the oscillation amplitude would not differ significantly from the free-running values, so the differential equation for the phase θ_n would become

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = \omega_n - \frac{\omega_n}{2Q}\sum_{m=1}^{N}\varepsilon_{nm}\frac{\alpha_{m0}}{\alpha_{n0}}\sin\left(\Phi_{nm} + \theta_n - \theta_m\right),\quad n = 1, 2, \ldots, N \tag{2}$$

when the locking occurs, all the oscillators will oscillate at the same frequency. Let us denote this common frequency as ω_{com}, we have

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = \omega_{\text{com}},\quad n = 1, 2, \ldots, N \tag{3}$$

Substituting in (2) we obtain

$$\omega_{\text{com}} = \omega_n\left[1 - \frac{1}{2Q}\sum_{n=1}^{N}\varepsilon_{nm}\frac{\alpha_{0m}}{\alpha_{0n}}\sin\left(\Phi_{nm} + \theta_n - \theta_m\right)\right],\quad n = 1, 2, \ldots, N \tag{4}$$

Choosing the phase of one oscillator as the reference, there are N unknowns including $(N-1)$ phases and ω_{com} itself. We can find these N unknowns by solving the N equations

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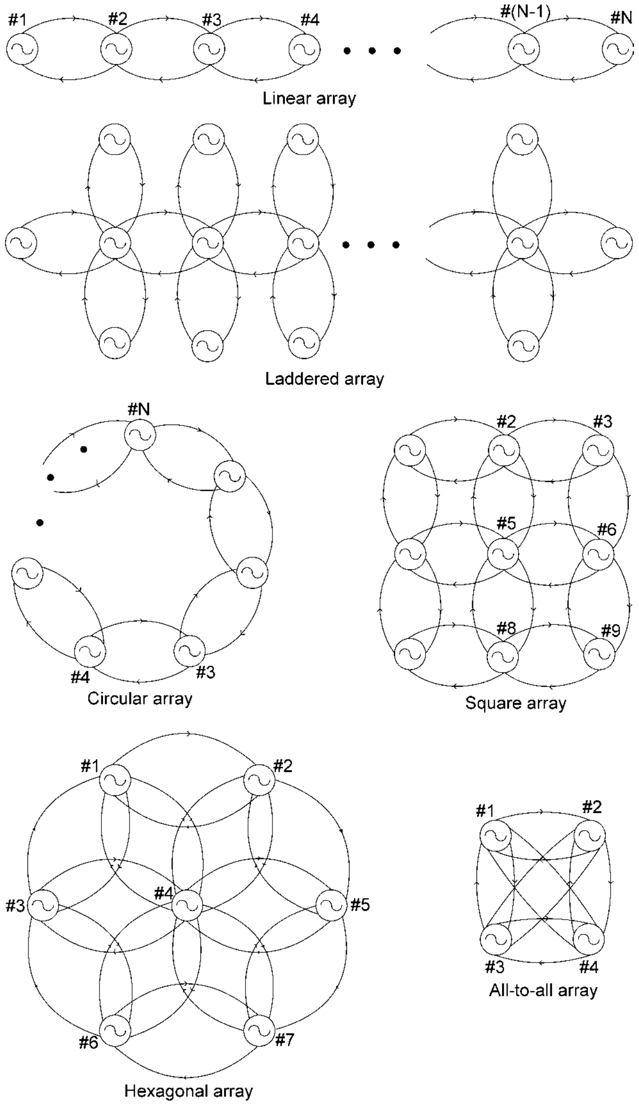

Fig. 1 *Different arrangements of the coupling networks*

in (4). However, there is no analytical solution for (2) or (4) and they must be solved numerically. As an example, consider a linear array of ten coupled oscillators. The free-running frequencies of oscillation are distributed normally around 1 GHz. The other parameters for this array are $Q = 15$, amplitude of their oscillation is 1 Ampere and the coupling factor is $0.2e^{j2\pi}$. $\Phi_{nm} = 2\pi$ is used because it results in in-phase oscillation, when the oscillators are similar. The differential equations of (2) are solved by the Runge–Kutta [8] method. The initial phase of each oscillator is set to a random value between $[-\pi, \pi]$ for each run. Figs. 3 and 4 show the locked and unlocked behaviour of coupled oscillators, respectively, for two different set of free-running frequency distributions. When the oscillators are locked once, they oscillate at the same frequency; note that constant phase differences between them are permissible. Fig. 3 shows locked oscillators but in Fig. 4 locking has not been achieved because of greater standard deviation of free-running frequencies.

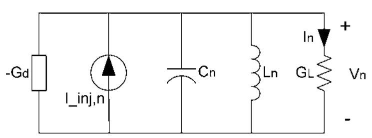

Fig. 2 *Shunt model for each oscillator*

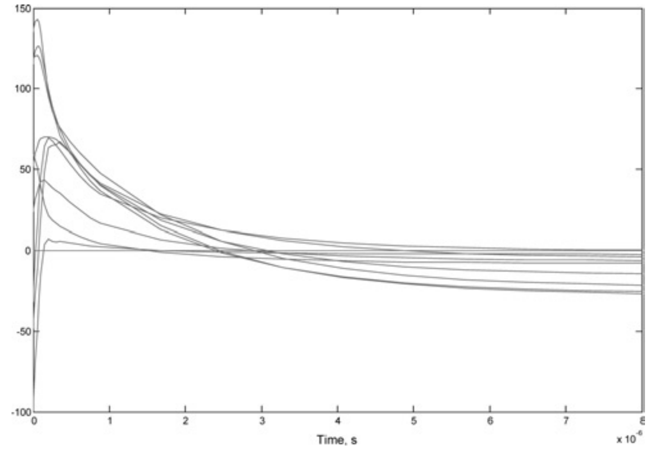

Fig. 3 *Relative phases (degree) of oscillators when locking is possible, $N = 10$, $f_0 = 1$ GHz, $Q = 15$, $\sigma_f = 1$ MHz*

Inter-injection locking of the oscillators is possible when there exists a set of solutions for θ_n in (4). In the previous work [6], the worst-case distribution of the oscillators' free-running frequencies was considered for which locking is possible and the locking bandwidth defined accordingly.

3 Statistical definition of locking bandwidth

Consider an array of N oscillators with given coupling coefficient between elements and given quality factor for each element in a specific configuration. As stated above, for a given distribution of free-running frequencies, the existence of a solution for θ_n means the possibility of locking. Assume that the free-running frequencies are Gaussian random variables with given mean ω_0 and variance σ_f^2

$$\omega_n = N\left(\omega_0, \sigma_f^2\right), \quad n = 1, 2, \ldots, N \tag{5}$$

For this distribution, the common frequency of oscillation would be ω_0 [7]. This can be deduced by adding equations (4) when the phase coupling is a multiple of 2π for similar oscillators. We define the locking bandwidth as the maximum standard deviation of free-running frequencies for which locking occurs with a probability more than a given value, for example, 0.9.

To compute this statistical locking bandwidth numerically, first, a program solves the set of differential equations

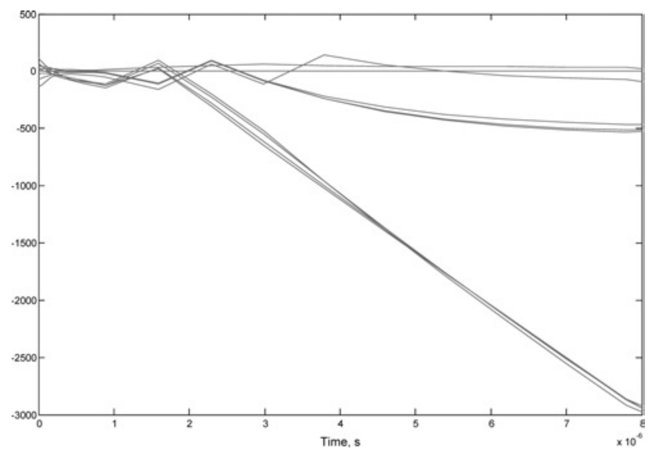

Fig. 4 *Relative phases (degree) of oscillators for unlocked condition, $N = 10$, $f_0 = 1$ GHz, $Q = 15$, $\sigma_f = 5$ MHz*

several hundred times each, with normally distributed free-running frequencies (with given mean and variance). At the end of each solution, another program determines whether the array is locked or not. In fact, this program determines if the relative phases of the oscillators remain constant or not. If they are constant, the output of this program is 1(locked) otherwise it is 0(unlocked). PL is obtained as the ratio of the locked cases to the total runs. Studying the various factors affecting the locking bandwidth is useful. These factors are the standard deviation of free-running frequencies, the quality factor of oscillators, the number of oscillators in the array, the coupling coefficient and also the coupling network topology. For a linear array, the locking bandwidth can be obtained analytically and the results compared. First, we study the influence of the variance of frequency distribution. By increasing the standard deviation of free-running frequencies, the PL will be decreased. For a linear array with ten elements, the following parameters have been chosen and the PL is obtained numerically

$$f_0 = 1\,\text{GHz}, \quad Q = 15, \quad \kappa = 0.2\text{e}^{j2\pi} \quad \text{and} \quad N = 10$$

As depicted in Fig. 5*a*, the result confirms the above statement. In contrast, for larger quality factors, it would be more difficult to change the frequency of an oscillator which is under injection. So, we expect that increasing the quality factor will result in decreasing the PL. The same linear array with $\sigma_f = 3$ MHz is considered and the PL is computed as a function of Q. In Fig. 5*b*, the PL is shown as a function of Q. By increasing the magnitude of the coupling factor, the injection to oscillator increases which increase the PL. Fig. 5*c* shows the PL in terms of coupling factor for the same array with $\sigma_f = 3$ MHz and $Q = 15$. It is obvious that increasing the number of oscillators will decrease the PL. This is shown in Fig. 5*d* for the same array. Finally, we study the effect of the coupling network topology on the PL. First, the three linear, circular and ladder networks are compared. In this study, all the other parameters of the networks are fixed and chosen as follows

$$f_0 = 1\,\text{GHz}, \quad Q = 15, \quad \kappa = 0.2\text{e}^{\,j2\pi} \quad \text{and} \quad N = 8$$

The probabilities of locking as a function of standard deviation are plotted in Fig. 6 which shows that the locking bandwidth for a circular array is more than that of the ladder and linear arrays. For example, the maximum standard deviation in which the PL is more than 90% is 3.5, 2.7 and 2.2 MHz for the circular, ladder and linear arrays, respectively. In Fig. 7, the same comparison was made between linear, circular and square arrays with nine elements. In Fig. 8, hexagonal and square networks of 37 and 36 elements are compared, respectively. The all-to-all and square arrays with $N = 9$ are compared in Fig. 9. From these simulations, it can be concluded that establishment of more coupling between elements of an array results in more locking bandwidth.

4 Analytical derivation of locking bandwidth for linear array

In [6], the worst-case distribution of free-running frequencies was used to compute the locking bandwidth and it was shown that locking bandwidth for circular arrays is twice that of linear arrays. In this paper, we assume that the free-running frequencies are normal random variables and we will obtain the probability density function (PDF) of locking in each array. The PL will be obtained by

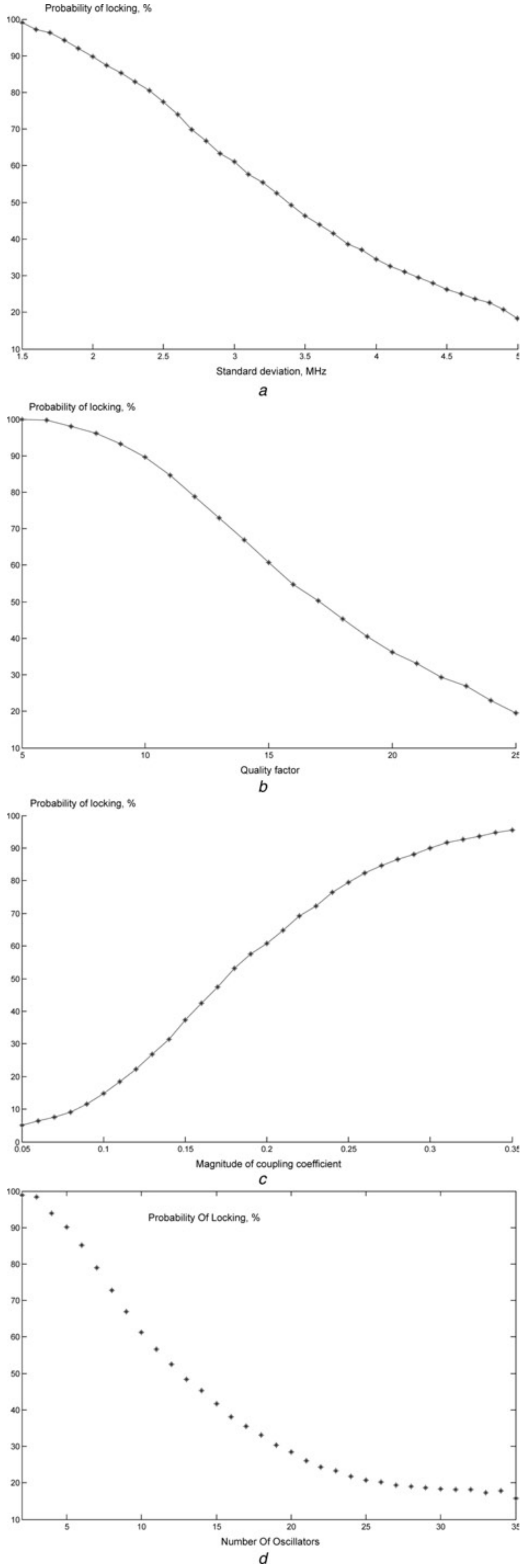

Fig. 5 *PL in terms of*

a Standard deviation of free-running frequencies
b Quality factor
c Coupling factor
d Number of oscillators

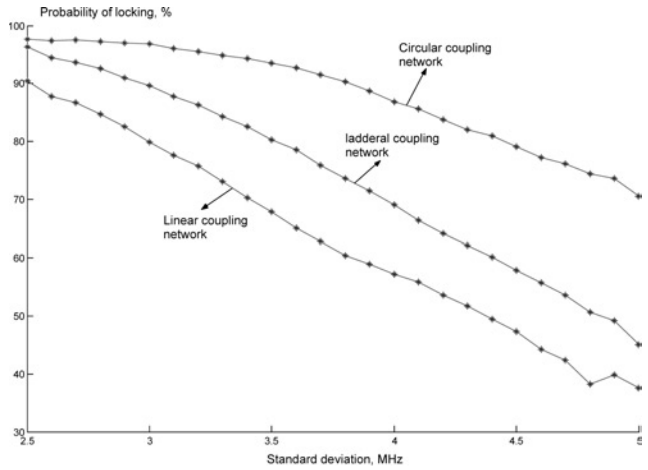

Fig. 6 *PL for linear, circular and ladder networks of eight elements,* $f_0 = 1$ *GHz,* $Q = 15$*,* $\kappa = 0.2e^{j2\pi}$

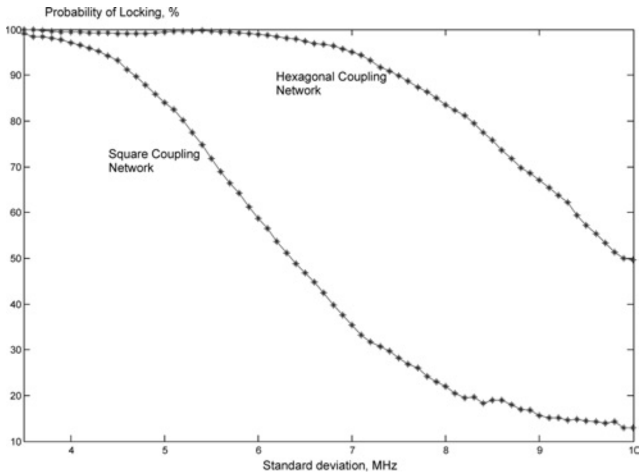

Fig. 8 *PL for hexagonal and square networks of 37 and 36 elements respectively,* $f_0 = 1$ *GHz,* $Q = 15$*,* $\kappa = 0.2e^{j2\pi}$

integrating this PDF. Consider a linear weakly coupled oscillator array, in which coupling factors are similar ($\varepsilon_{ij} = \varepsilon$), and $\varepsilon\,\omega_i/2Q$ is approximately equal for all elements because ω_n have very little difference; so, we define

$$\alpha = \frac{\varepsilon\,\omega_0}{2Q} \tag{6}$$

In linear arrays, we have

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}t} = \omega_n - \alpha \sum_{\substack{m = n-1 \\ m \neq n}}^{n+1} \sin(\Phi_{nm} + \theta_n - \theta_m),$$

$$n = 1, 2, \ldots, N \tag{7}$$

Subtracting two consecutive equations of (7) and by assuming $\Phi = 2k\pi$, one obtains [7]

$$\frac{\mathrm{d}\Delta\theta_n}{\mathrm{d}t} = \Omega_n + \alpha\left(\sin(\Delta\theta_{n-1}) - 2\sin(\Delta\theta_n) + \sin(\Delta\theta_{n+1})\right),$$

$$n = 2, \ldots, N-1 \tag{8}$$

in which

$$\Delta\theta_n = \theta_n - \theta_{n+1} \quad \text{and} \quad \Omega_n = \omega_n - \omega_{n+1}$$

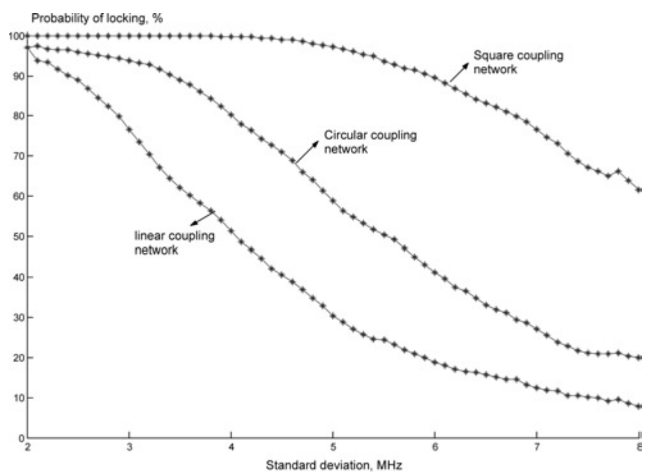

Fig. 7 *PL for linear, circular and square networks of nine elements,* $f_0 = 1$ *GHz,* $Q = 15$*,* $\kappa = 0.2e^{j2\pi}$

$\Phi = 2k\pi$ was chosen since an array of similar oscillators oscillate in phase and power combining is achievable in the boresight. After the locking, the time variation of phase differences will be zero, so (8) reduces to

$$\alpha[\boldsymbol{A}]\overline{\boldsymbol{S}} = -\overline{\boldsymbol{\Omega}} \tag{9}$$

where $\overline{\boldsymbol{S}} = \left[\sin\Delta\theta_1 \quad \sin\Delta\theta_2 \quad \ldots \quad \sin\Delta\theta_{N-1}\right]^{\mathrm{T}}$ and $\overline{\boldsymbol{\Omega}} = \left[\Omega_1 \quad \Omega_2 \quad \ldots \quad \Omega_{N-1}\right]^{\mathrm{T}}$

$$[\boldsymbol{A}] = \begin{bmatrix} -2 & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & 1 & -2 & 1 \\ 0 & 0 & \ldots & \ldots & 0 & 1 & -2 \end{bmatrix}_{(N-1)\times(N-1)} \tag{10}$$

and $\overline{\boldsymbol{S}}$ is obtained from

$$\overline{\boldsymbol{S}} = -\frac{1}{\alpha}\left[A^{-1}\right] \quad \overline{\boldsymbol{\Omega}} \tag{11}$$

Locking is possible when there exists a valid solution for $\overline{\boldsymbol{S}}$, that is, the absolute value of all the members of $\overline{\boldsymbol{S}}$ are less

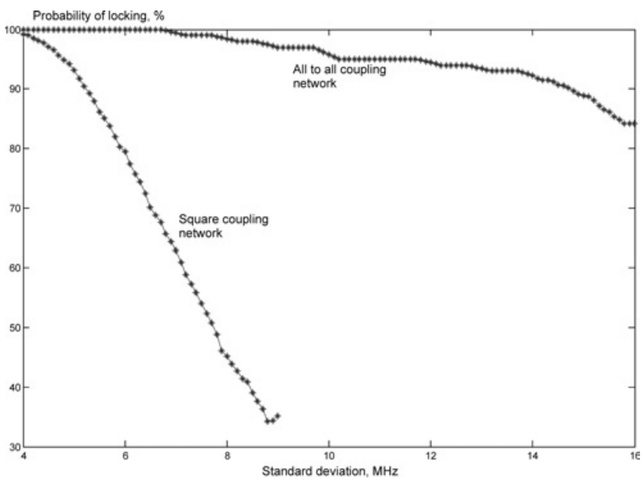

Fig. 9 *PL for all-to-all and square networks with nine elements,* $f_0 = 1$ *GHz,* $Q = 15$*,* $\kappa = 0.2e^{j2\pi}$

than or equal to unity. Elements of $[A^{-1}]$ are [9]

$$[A^{-1}]_{nm} = a_{nm} = \frac{m(n-N)}{N}, \quad n \geq m$$

$$[A^{-1}]_{mn} = [A^{-1}]_{nm} \tag{12}$$

Re-sorting (11) in terms of ω_n, one obtains

$$\bar{S} = [B]_{(N-1)\times N} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \cdots \\ \omega_n \end{bmatrix}_{N\times 1} \tag{13}$$

and $[B]$ has the following form

$$[B] = \frac{1}{\alpha} \begin{bmatrix} \frac{N-1}{N} & -\frac{1}{N} & \cdots & \cdots & \cdots & -\frac{1}{N} \\ \frac{N-2}{N} & \frac{N-2}{N} & -\frac{2}{N} & \cdots & \cdots & -\frac{2}{N} \\ \frac{N-3}{N} & \frac{N-3}{N} & \frac{N-3}{3} & -\frac{3}{N} & \cdots & -\frac{3}{N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{N} & \frac{1}{N} & \cdots & \cdots & \frac{1}{N} & -\frac{(N-1)}{N} \end{bmatrix}_{(N-1)\times(N)} \tag{14}$$

The distribution of variables ω_n is assumed to be known and the PDF of S_n can be obtained from the following equation [10]

$$f(S_1, S_2, \ldots, S_{N-1}) = f(\bar{S}) = \frac{1}{\sqrt{(2\pi)^{N-1}\det[V]}} e^{(-1/2)(\bar{S}-\bar{\mu}))^{\mathrm{T}}\,[V^{-1}]\,(\bar{S}-\bar{\mu})} \tag{15}$$

in which $\bar{\mu}$ is the mean of S_n that is zero. The matrix $[V]$ is the covariance matrix of S_n.

$$[V] = \sigma_f^2 [B][B]^{\mathrm{T}} \tag{16}$$

The determinant of $[V]$ and $[V^{-1}]$ can be obtained from

$$\det([V]) = \frac{1}{N}\left(\frac{\sigma_f^2}{\alpha^2}\right)^{N-1} \tag{17}$$

$$[V^{-1}] = \frac{-\alpha^2}{\sigma_f^2} \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 1 & -2 & 1 \\ \cdots & \cdots & 0 & +1 & -2 \end{bmatrix} \tag{18}$$

So the PDF of S_n would be

$$f(\bar{S}) = \frac{1}{\sqrt{(2\pi)^{N-1}(1/N)(\sigma^2/\alpha^2)^{N-1}}} \times e^{-(\alpha^2/\sigma^2)(s_1^2 + \cdots + s_{N-1}^2 - s_1 s_2 - \cdots - s_{N-2}s_{N-1})} \tag{19}$$

The results are justified by what has been obtained in [11], in which it was shown that for linear arrangement of oscillators, the PL is zero, when N increases infinitely.

The PL will be obtained by integrating (19) in $(N-1)$ dimension unity cube so that S_n with absolute values less than or equal to unity can be obtained.

Regarding the stability of the oscillator array, it should be noted that once all the elements of vector $\bar{S}$ have a value of less than unity, the system would lock and the phases tend to a stable value [12].

Now consider a linear array with the following parameters

$$f_0 = 1\,\text{GHz}, \quad Q = 10, \quad \kappa = 0.15\mathrm{e}^{\mathrm{j}2\pi} \quad \text{and} \quad N = 10$$

We obtain the PL by integrating the PDF in (19) and compare it with the numerical simulation as earlier. Since the integrand is a 9-variable function, integrating it with common numerical methods is time-consuming and sometimes impossible. So the Gaussian-quadrature [13] method has been used for this purpose. The results are shown in Fig. 10, was it can be seen that the difference between the two curves is insignificant. The most important source of error is numerical integration.

5 PL in ladder array

Similarly, with the assumption of weak coupling, we consider only the differential equations for phases. For ladder arrays, the numbering of oscillators is important. If the numbering is done well (like Fig. 11), the matrix computations will be easier.

By defining $s_{ij} = \sin(\theta_i - \theta_j)$ and $\Omega_i = \omega_i - \omega_{i+1}$, (11) is obtained for the ladder array in which $[A]$ has the following form

$$[A] = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 2 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -1 & 2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & -1 & 1 & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & -1 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & -1 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{(N-1)(N-1)} \tag{20}$$

By rearranging (11) in terms of ω_n, we obtain an equation similar to (15) for the ladder array, in which $[B]$ as in

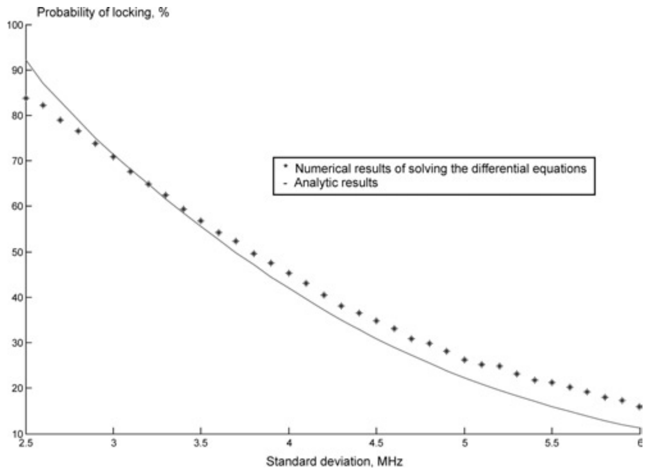

Fig. 10 *PL for a linear array obtained by two different methods*
Analytical and numerical simulation

(13) has the following form

$$[\boldsymbol{B}] = \frac{-1}{\alpha}\begin{bmatrix} \frac{N-1}{N} & \frac{-1}{N} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \frac{-1}{N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 & \frac{-(N-1)}{N} & 1 & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & 1 & 1 & \frac{-(N-1)}{N} & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \frac{-(N-1)}{N} & 1 \\ 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 1 & \frac{-(N-1)}{N} \end{bmatrix}_{(N-1)\times(N)} \tag{21}$$

We can obtain the PDF of S_n from (15). Now the matrix $[\boldsymbol{V}^{-1}]$ has the following form

$$[\boldsymbol{V}^{-1}] = \frac{\alpha^2}{\sigma_f^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 2 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -1 & 2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 2 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 2 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 2 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & 2 \end{bmatrix} \tag{22}$$

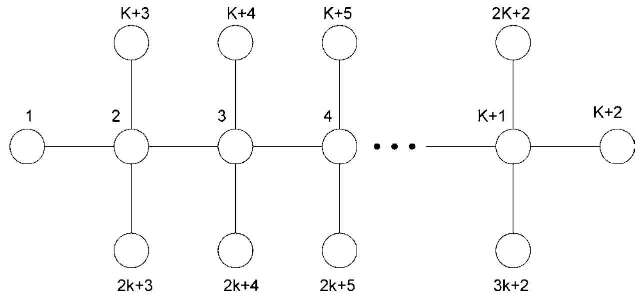

Fig. 11 *Ladder array*

and the determinant of $[\boldsymbol{V}]$ is obtained from

$$\det([\boldsymbol{V}]) = \frac{1}{N} \times \left(\frac{\sigma_f^2}{\alpha^2}\right)^{N-1} \tag{23}$$

There is a hint which can help us in computing the exponential term of the PDF. The exponential term of the PDF for an N-element ladder array has $(3N-5)$ terms. $(N-1)$ of them are for branches of the graph, which are squared. Others are for neighbour steps. For these terms, if a step has a neighbour with the preceding step, the sign of that term will be negative, but if a step has a neighbour with the next step, the sign of that term will be positive. The next example illustrates this. Consider a 3-step ladder array with 11 elements. The exponential term has $(3\times 11-5=28)$ terms. Ten of them are for the branches of the graph which are squared:

$$s_{21}^2 + s_{32}^2 + s_{43}^2 + \cdots + s_{11,4}^2$$

Six terms are for the first step

$$s_{21}s_{32} + s_{92}s_{32} + s_{62}s_{21} + s_{62}s_{92} + s_{21}s_{92} + s_{32}s_{62}$$

All signs of these terms are positive, because there is no term between this step and the last.

Six terms are for the second and six for the third step

$$s_{43}s_{73} + s_{10,3}s_{43} + s_{10,3}s_{73} - s_{73}s_{32} - s_{32}s_{43} - s_{10,3}s_{32}$$
$$s_{84}s_{11,4} + s_{84}s_{54} + s_{11,4}s_{54} - s_{43}s_{54} - s_{84}s_{43} - s_{11,4}s_{43}$$

As it can be seen, three terms of each two steps has a negative sign, because they are between that step and the last.

Consider a ladder array with the parameters

$$f_0 = 1\,\text{GHz}, \quad Q = 10, \quad \kappa = 0.15\text{e}^{\text{j}2\pi} \quad \text{and} \quad N = 5$$

The numerical solution of the differential equations and integration of the PDF are compared in Fig. 12. There is a very slight difference between the two methods.

Till now, the PL were obtained for linear and ladder arrays. However, for other configurations, the PDF cannot be obtained analytically. For example, consider a square array with nine elements. The steady-state solution of (8)

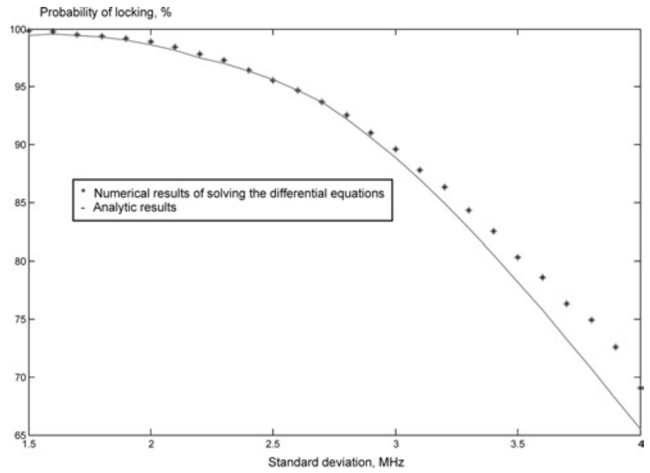

Fig. 12 *PL for ladder array with* $N = 5$, $f_0 = 1$ *GHz*, $Q = 10$, $\kappa = 0.15e^{j2\pi}$, *numerical and analytical results*

results in

$$\begin{aligned}
&\omega_1 + \alpha(s_{21} + s_{31}) = 0\\
&\omega_2 + \alpha(-s_{21} + s_{42} + s_{52}) = 0\\
&\omega_3 + \alpha(-s_{31} + s_{53} + s_{63}) = 0\\
&\omega_4 + \alpha(-s_{42} + s_{74}) = 0\\
&\omega_5 + \alpha(-s_{52} - s_{53} - s_{75} - s_{85}) = 0\\
&\omega_6 + \alpha(-s_{63} + s_{86}) = 0\\
&\omega_7 + \alpha(-s_{74} - s_{75} + s_{97}) = 0\\
&\omega_8 + \alpha(-s_{85} - s_{86} + s_{98}) = 0\\
&\omega_9 + \alpha(-s_{97} - s_{98}) = 0
\end{aligned} \tag{24}$$

The matrix form of these equations is

$$\alpha \begin{bmatrix} 2 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} s_{21}\\ s_{31}\\ s_{42}\\ s_{52}\\ s_{53}\\ s_{63}\\ s_{74}\\ s_{75}\\ s_{85}\\ s_{86}\\ s_{97}\\ s_{98} \end{bmatrix} = - \begin{bmatrix} \Omega_1\\ \Omega_2\\ \Omega_3\\ \Omega_4\\ \Omega_5\\ \Omega_6\\ \Omega_7\\ \Omega_8 \end{bmatrix} \tag{25}$$

We have eight equations and 12 unknowns; the other equations are

$$\begin{aligned}
&\theta_{97} + \theta_{75} + \theta_{58} + \theta_{89} = 0\\
&\theta_{85} + \theta_{53} + \theta_{36} + \theta_{68} = 0\\
&\theta_{74} + \theta_{42} + \theta_{25} + \theta_{57} = 0\\
&\theta_{52} + \theta_{21} + \theta_{13} + \theta_{35} = 0
\end{aligned} \tag{26}$$

θ_{mn} means $\theta_{mn} = \theta_m - \theta_n$. These equations are based on the fact that the phase difference in a closed loop is zero. If we rearrange these equations in terms of S_n, we obtain

$$\begin{aligned}
&\sin^{-1}(s_{97}) + \sin^{-1}(s_{75}) + \sin^{-1}(s_{58}) + \sin^{-1}(s_{89}) = 0\\
&\sin^{-1}(s_{85}) + \sin^{-1}(s_{53}) + \sin^{-1}(s_{36}) + \sin^{-1}(s_{68}) = 0\\
&\sin^{-1}(s_{74}) + \sin^{-1}(s_{42}) + \sin^{-1}(s_{25}) + \sin^{-1}(s_{57}) = 0\\
&\sin^{-1}(s_{52}) + \sin^{-1}(s_{21}) + \sin^{-1}(s_{13}) + \sin^{-1}(s_{35}) = 0
\end{aligned} \tag{27}$$

These equations are nonlinear and they cannot be solved analytically, and hence we cannot obtain the PDF of S_n analytically. It can be concluded that the PDF an be obtained analytically when there is no closed loop in the graph of an array. In fact, we can only obtain the PDF analytically, for arrays the graphs of which include trees (with no closed loop). The linear and ladder arrays have this property.

6 Conclusion

The statistical locking bandwidth of an array of mutually coupled oscillators was investigated and it was computed by solving the governing differential equations numerically. The effects of various factors in the PL were studied. These factors were the standard deviation of free-running frequencies, the quality factor of oscillators, the number of oscillators in the array, the coupling coefficient and more significantly the coupling network topology. The analytical locking bandwidth was derived for linear and ladder arrays and the PL was calculated for these configurations by the integration of PDFs and the results were compared with the numerical simulations. The PL objectively shows the expected locking bandwidth in arrays of mutually coupled oscillators with the normal distribution of free-running frequencies in different topologies. A reasonable estimate was obtained in each case.

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