Recovery of Signals from Nonuniform Samples Using Iterative Methods

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***Abstract*—In this paper, we propose an iterative method to recover a band-limited signal from its ideal nonuniform samples. The convergence of iterations is proved and general regions for convergence are found. It is shown that the iterative method is also applicable to other forms of nonuniform sampling, i.e., natural sampling and interpolated sampling (such as sample-and-hold signal).**

I. Introduction

Nonuniform sampling has many applications in signal processing; for a review of these applications, see [1]. The reconstruction of a signal from nonuniform samples is not a trivial task, however; the most promising method is the iterative method. The application of iterative methods based on Sandberg theorem [2] was first used in nonuniform sampling or FM zero crossings by Wiley *et al.* [3] and [4]. Wiley *et al.* used pulses of finite width at instants not exactly at nonuniform instants $\{t_k\}$ but rather "smeared" or averaged samples over a finite support. They conjectured, but did not prove, that the iterative method is valid for ideal impulsive nonuniform samples. Marvasti [5] proved that the iterative method converges for random samples when the samples are chosen from a uniform or a Poisson distribution. Sauer *et al.* [6] extended the work of [3] to 2-D signals. A summary of all the previous works is given in [1]. It is the objective of this paper to prove that iterative methods can be used for the recovery of band-limited signals from ideal impulsive nonuniform samples. We also show that the method works for natural nonuniform samples as well as the sampled-and-held version of nonuniform instants $\{t_k\}$.

II. Recovery of Band-Limited Signals from Impulsive Nonuniform Samples

We desire to show that, given the nonuniform samples $x_s(t) = \Sigma_i\, x(t_i)\, \delta(t - t_i)$ where $\{t_k\}$ is a stable sampling set,[1] the following iterative method shall recover the band-limited finite energy signal $\boldsymbol{x}(t)$ from $x_s(t)$, i.e.,

$$x_{k+1}(t) = \lambda PSx(t) + (P - \lambda PS)\, x_k(t) \tag{1}$$

where λ, $\boldsymbol{x}(t)$ and $x_k(t)$ are a convergence constant, the original finite energy signal and the kth iteration, respectively. P and S are, respectively, the band-limiting and ideal nonuniform sampling operators. $PS\boldsymbol{x}(t)$ in (1) is the low-pass filtered version of the ideal nonuniform samples, which is known. In the following, we are going to prove that there exists a range for λ for which the process will converge to a stable point, which is equal to the designed signal $\boldsymbol{x}$, i.e.,

$$\lim_{k\to\infty} x_k(t) \to x(t).$$

Proof: The iteration will converge: $\{\lim_{k\to\infty} x_k \to x(t)\}$ if $P - \lambda PS$ is a contraction. The operator $P - \lambda PS$ is a contraction [6] if

$$\left\|x^{(k+1)}(t)\right\| \le \left\|x^{(k)}(t)\right\| \qquad \text{for all } k \tag{2}$$

where $\|\boldsymbol{x}(t)\|$ is the L^2 norm of $\boldsymbol{x}(t)$ and $x^{(k)}(t) = x_k(t) - x_{k-1}(t)$ or from (1), we have

$$\left\|P(x^{(k)}(t)) - \lambda PS(x^{(k)}(t))\right\| \le r\left\|x^{(k)}(t)\right\| \tag{3}$$

where $0 \le r < 1$.

So if (3) is satisfied, the theorem is proved. We now show that there is a λ such that (3) is true. The left-hand side of (3) can be written as

$$\begin{aligned}\left\|P(x^{(k)}(t)) - \lambda PS(x^{(k)}(t))\right\|^2 &= \left\|x^{(k)}(t)\right\|^2 + \lambda^2\left\|PS(x^{(k)}(t))\right\|^2 \\ &\quad - 2\lambda \int P(x^{(k)}(t))\, PS(x^{(k)}(t))\, dt.\end{aligned} \tag{4}$$

Now, we show that there is a positive real k_1 and k_2 such that

$$\int P(x^{(k)}(t))\, PS(x^{(k)}(t))\, dt \ge k_1\left\|x^{(k)}(t)\right\|^2 \tag{5}$$

and

$$\left\|PS(x^{(k)}(t))\right\|^2 \le k_2\left\|x^{(k)}(t)\right\|^2. \tag{6}$$

To prove (5), we first note that

$$\int P(x^{(k)}(t))\, PS(x^{(k)}(t))\, dt = \int (x^{(k)}(t))\, S(x^{(k)}(t))\, dt. \tag{7}$$

Since the operator P is self-adjoint (i.e, $\int (Px)\, y\, dt = \int x(Py)\, dt$), $P^2 = P$, and $Px_k = x_k$. Now, (7) can be written as

$$\int (x^{(k)}(t))\, S(x^{(k)}(t))\, dt = \sum_i \left[x^{(k)}(t_i)\right]^2. \tag{8}$$

Since the instants $\{t_k\}$ are assumed to be a stable sampling set, from [7] or [1] we have for all the signals $x(t)$ that are band limited

$$A \le \frac{\sum_i x^2(t_i)}{\left\|x(t)\right\|^2} \le B \tag{9}$$

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[1] A stable sampling set is a set that uniquely determines a band-limited signal and $\int |x(t)|^2\, dt \le C\, \Sigma_{n=-\infty}^{\infty} |x(t_n)|^2$, where C is a constant. For instance, if $|t_k - kT| < (T/4)$, the set is a stable sampling set. If the samples are all clustered in a finite interval, the sampling set uniquely determines the signal but is not stable. For more information, see [1].

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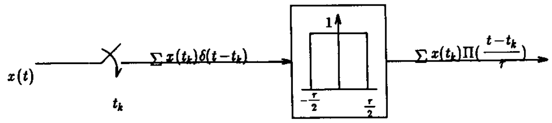

Fig. 1.

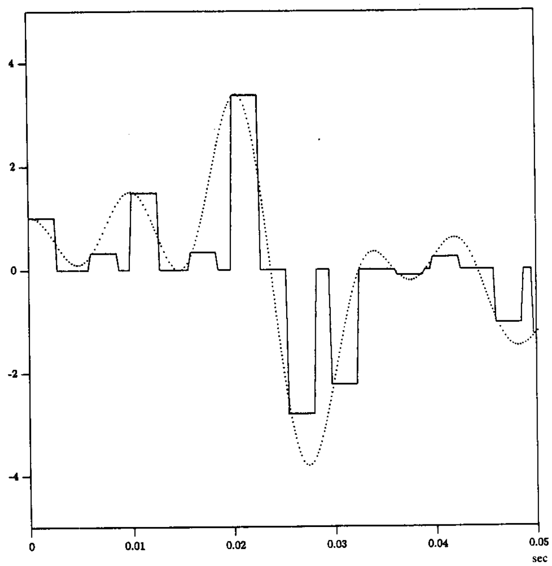

Fig. 2. Sample and hold with constant width at the Nyquist rate ($|t_n - nT| < T/4$). Original signal: · · · · · · · · . Sample and hold with constant width: ——.

where A and B are positive constants, depending only on the bandwidth and $\{t_k\}$. From (7)–(9), we conclude

$$\int P(x^{(k)}(t))\, PS(x^{(k)}(t))\, dt \geq A\left\|x^{(k)}(t)\right\|^2. \tag{10}$$

Therefore, (5) is proved by setting $k_1 = A$.

To prove (6), we can write

$$\begin{aligned}\left\|PS(x^{(k)}(t))\right\|^2 &= \int PS(x^{(k)}(t)) \sum_i (x^{(k)}(t))\, \delta(t - t_i)\, dt\\ &= \sum_i \int \left[PS(x^{(k)}(t))\right] (x^{(k)}(t_i)\, \delta(t - t_i))\, dt\\ &= \sum_i \left[PS(x^{(k)}(t))\right] t = t_i\, (x^{(k)}(t_i))\\ &\leq \left[\sum_i \left|x^{(k)}(t_i)\right|^2\right]^{1/2} \left[\sum_i \left|\left[PS(x^{(k)}(t))\right] t = t_i\right|^2\right]^{1/2}\\ &\leq B\left\|x^{(k)}(t)\right\| \cdot \left\|PS(x^{(k)}(t))\right\|.\end{aligned} \tag{11}$$

Hence, from (11), we have

$$\left\|PS(x^{(k)}(t))\right\| \leq B\left\|x^{(k)}(t)\right\|. \tag{12}$$

Therefore, hypothesis (6) is proved. From (4)–(6), we get

$$\left\|P(x^{(k)}(t)) - \lambda PS(x^{(k)}(t))\right\|^2 \leq r\left\|x^{(k)}(t)\right\|^2$$

where $r = 1 + \lambda^2 k_2 - 2\lambda k_1$.

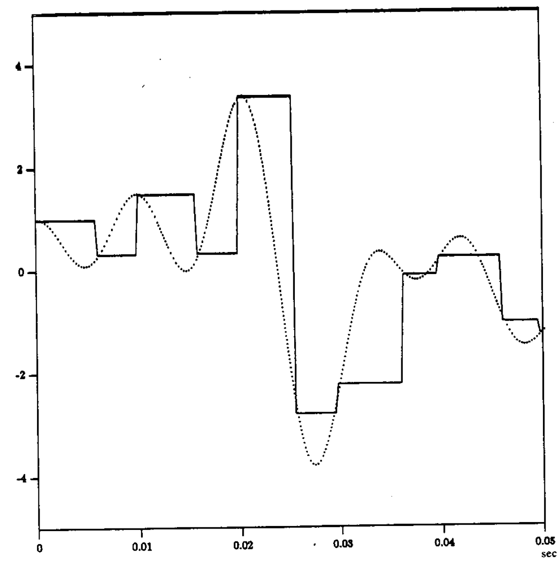

Fig. 3. Sample and hold with variable width at the Nyquist rate ($|t_n - nT| < T/4$). Original signal: · · · · · · · · . Sample and hold with variable width at the Nyquist rate: ——.

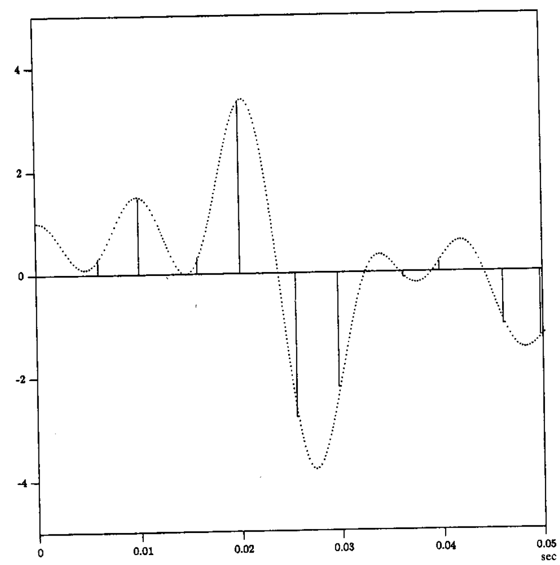

Fig. 4. Original signal and nonuniform ideal samples at the Nyquist rate ($|t_n - nT| < T/4$). Original signal: · · · · · · · · . Nonuniform ideal samples: ——.

In order to satisfy (3), r has to be in $0 \leq r < 1$, i.e., given a particular k_1 and k_2, λ has to be in the following region of convergence for the iterative relationship given in (1):

$$\begin{cases} 0 < \lambda < \dfrac{2k_1}{k_2} \\ k_1^2 \leq k_2. \end{cases} \tag{13}$$

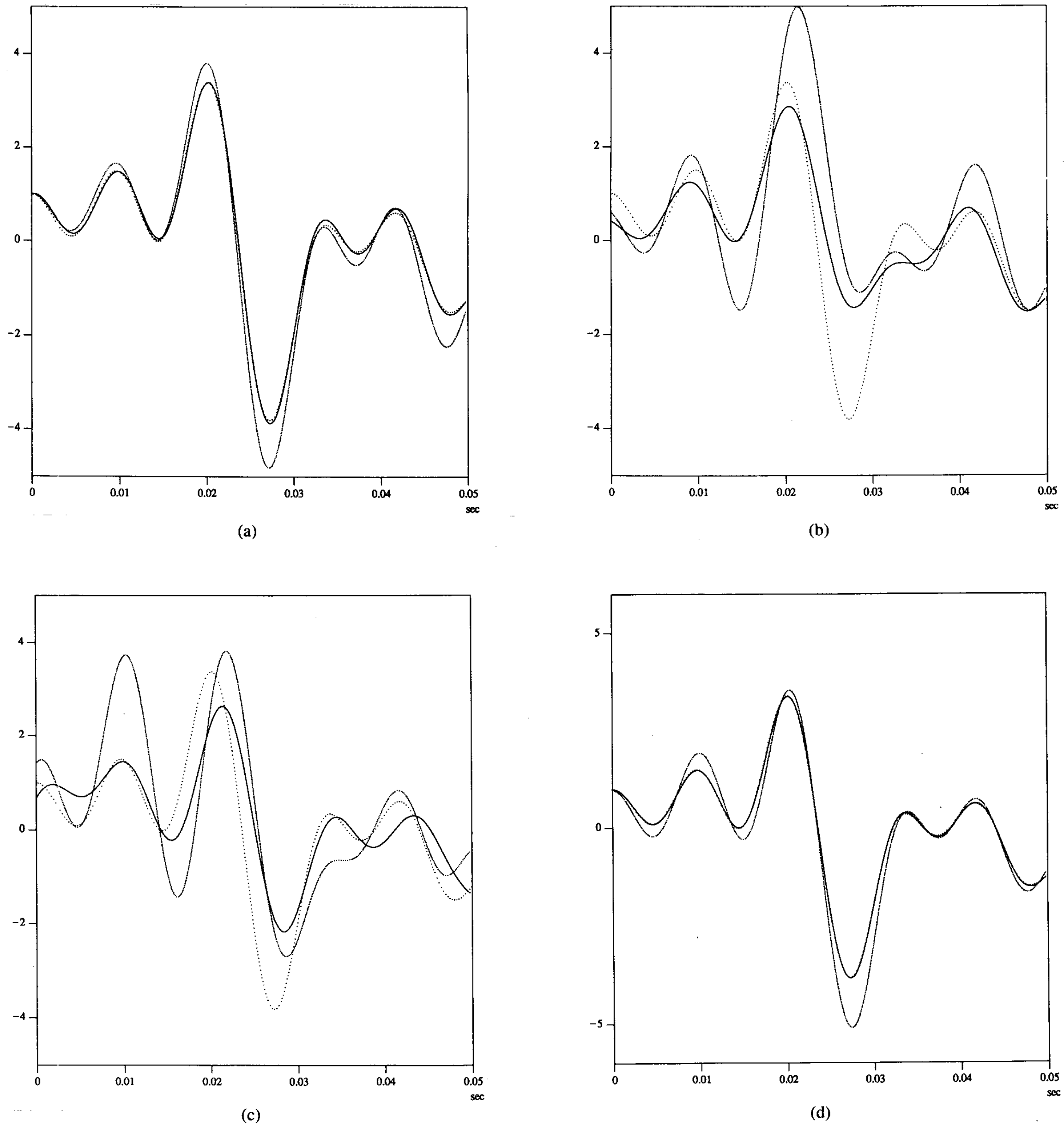

Fig. 5. Reconstruction from ideal nonuniform samples at the Nyquist rate ($|t_n - nT| < T/2$). (b) Reconstruction from ideal nonuniform samples at the Nyquist rate ($|t_n - nT| < T/2$). (c) Reconstruction from ideal nonuniform samples at the Nyquist rate (random samples). (d) Reconstruction from ideal nonuniform samples at twice the Nyquist rate ($|t_n - nT| < T/4$). (Continued on next page.)

Notice that to minimize r and hence maximize the convergence speed, it is required that $\lambda = (k_1/k_2)$. Also note that (13) is not limited to ideal sampling but rather is true for any distortion operator S and constraint operator P that satisfy (3). The main problem in choosing a proper λ is that it cannot be determined theoretically since k_1 and k_2 are not known. Therefore, experimental results will determine the range of λ. In our simulation, depending on the nonuniform sampling set, λ is taken to be somewhere between 0.5 and 1.

III. Other Forms of Nonuniform Sampling

A. Natural Sampling

Natural samples are defined as the product of the band-limited signal by rectangular pulses that are nonuniformly spaced at instants $\{t_k\}$ and have the height of unity and a width of τ. If we find a k_1 and k_2 that satisfy (5) and (6), we can use the iterative method (1) to recover a band-limited finite energy signal from its nonuniform samples. Equation (5) similar to (8)

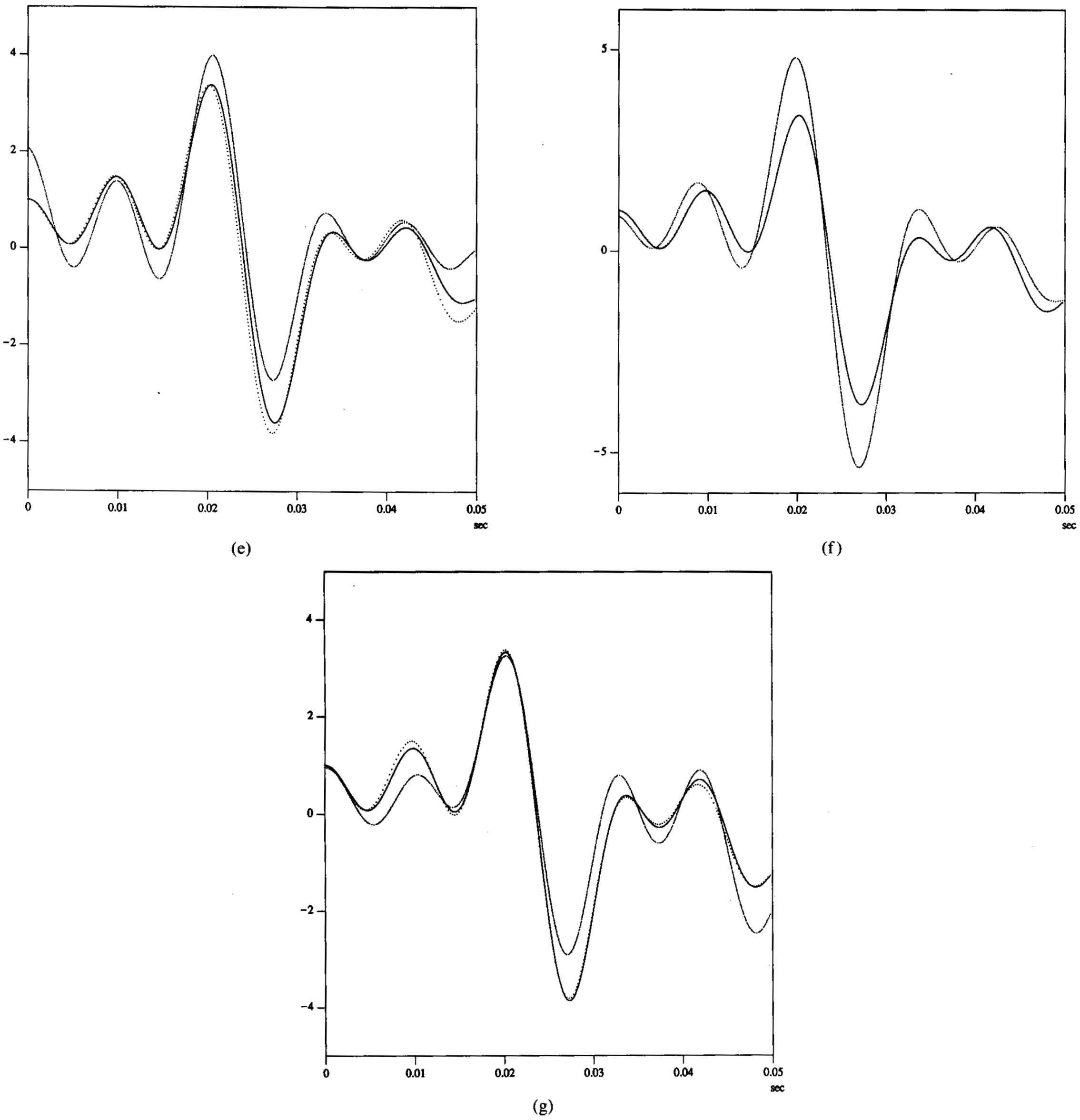

Fig. 5. (*Continued.*) (e) Reconstruction from ideal nonuniform samples at twice the Nyquist rate (random samples). (f) Reconstruction from ideal nonuniform samples at three times the Nyquist rate($|t_n - nT| < T/4$). (g) Reconstruction from ideal nonuniform samples at three times the Nyquist rate (random samples). Original signal: · · · · · · · · . Filtered samples: ·········. After 10 iterations: ——.

becomes

$$\int (x^{(k)}(t))\, S(x^{(k)}(t))\, dt$$

$$= \int S(x^{(k)}(t))^2\, dt \geq k_1 \left\| x^{(k)}(t) \right\|^2 \qquad (14)$$

for all k.

Equation (14) has a solution because $\int S(x^{(k)}(t))^2\, dt$ is a positive number[2] and so is $\|x^{(k)}(t)\|^2$. Therefore, there is a constant k_1 such that (14) is true for all values of k. Since $\int S(x^{(k)}(t))^2\, dt < \|x^{(k)}(t)\|^2$, k_1 is smaller than 1. Also, (6) is true because

$$\left\| PS(x^{(k)}(t)) \right\|^2 \leq \left\| S(x^{(k)}(t)) \right\|^2 < \left\| x^{(k)}(t) \right\|^2$$

so k_2 can be taken as 1. In this case the range of λ in (13) becomes $0 < \lambda < 2$.

B. Sample-and-Hold Interpolation

The sample-and-hold operation on a signal can be considered as a two-step procedure: 1) ideal nonuniform sampling and 2) passing the samples through a zero-order hold filter. Thus, the S operator in this case should be convolved by a zero-order hold impulse response (Fig. 1).

The width of zero-order-hold impulse response can be con-

[2] $\int S(x^{(k)}(t))^2\, dt$ cannot be zero since $S[x^{(k)}(t)]$ cannot be identically zero for a band-limited signal.

TABLE I
MEAN-SQUARE ERROR ($T/4$ INTERVAL) AT NYQUIST RATE

Iteration No.	Mean-Square Error: Variable S&H	Constant S&H	Ideal	Natural
0	1.864412e-1	4.365404e-1	1.580624e-1	5.428200e-1
1	2.684877e-2	9.017097e-2	2.535609e-3	1.208514e-1
2	4.488294e-3	2.132896e-2	1.746579e-3	2.655716e-2
3	1.975845e-3	6.128620e-3	1.607121e-3	6.746200e-3
4	2.217043e-3	2.903171e-3	1.575803e-3	3.343687e-3
5	2.644699e-3	2.641249e-3	1.522073e-3	3.220765e-3
6	2.940791e-3	3.070048e-3	1.475997e-3	3.551968e-3
7	3.122244e-3	3.567942e-3	1.439057e-3	3.817421e-3
8	3.233513e-3	3.977919e-3	1.411968e-3	3.975198e-3
9	3.305488e-3	4.280155e-3	1.392150e-3	4.059322e-3
10	3.356229e-3	4.491736e-3	1.378551e-3	4.101327e-3

TABLE II
MEAN-SQUARE ERROR FOR RANDOM SAMPLES

Iteration No.	Mean-Square Error: Nyquist Rate	2× Nyquist	3× Nyquist
0	1.672375e+0	4.450645e-1	2.719940e-1
1	1.143412e+0	2.491319e-1	9.645810e-2
2	9.669778e-1	1.789243e-1	4.881012e-2
3	9.370443e-1	1.405293e-1	2.707859e-2
4	8.414629e-1	1.131710e-1	1.639921e-2
5	8.303990e-1	9.394870e-2	1.104171e-2
6	7.694325e-1	7.923566e-2	8.194261e-3
7	7.596434e-1	6.719316e-2	6.596941e-3
8	7.190652e-1	5.758550e-2	5.629838e-3
9	7.089961e-1	4.926686e-2	4.992424e-3
10	6.810131e-1	4.243471e-2	4.530937e-3

TABLE III
MEAN-SQUARE ERROR AT $T/4$ INTERVAL

Iteration No.	Mean-Square Error: Nyquist Rate	2× Nyquist	3× Nyquist
0	1.580624e-1	1.905335e-1	4.7484000e-1
1	2.535609e-3	2.485873e-2	4.2737980e-2
2	1.746579e-3	5.125920e-3	1.2164360e-2
3	1.607121e-3	1.089340e-3	4.7587000e-3
4	1.575803e-3	7.573805e-4	2.139869e-3
5	1.522073e-3	5.363611e-4	1.1480990e-3
6	1.475997e-3	5.348017e-4	5.3484040e-4
7	1.439057e-3	4.930253e-4	4.4935560e-4
8	1.411968e-3	4.781214e-4	2.7248240e-4
9	1.392150e-3	4.545103e-4	3.1401990e-4
10	1.378551e-3	4.353163e-4	2.566236e-4

sidered to be either a constant (Fig. 2) or a variable (Fig. 3) depending on $t_k - t_{k-1}$.

To prove the convergence of the proposed iterative technique, we should show that there exists a unique fixed point. The set of sampled-and-held nonuniform samples is a convex set.[3] According to [8, theorem 2.3-3] the sample-and-hold op-

[3]A set C is said to be convex if together with any x_1 and x_2 in C, it also contains $\mu x_1 + (1 - \mu)x_2$ for any value of μ in $0 \leq \mu \leq 1$. In our case, the set is a set of functions for sample and hold or nth order hold polynomial.

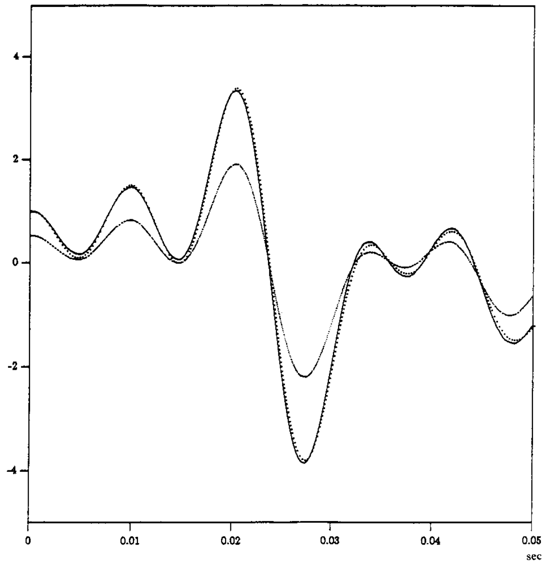

Fig. 6. Reconstruction from sample and hold with constant width at the Nyquist rate ($|t_n - nT| < T/4$). Original signal: ········. Filtered samples: ··········. After 5 iterations: ——.

eration is nonexpansive and there exists at least one fixed point for the iteration. Since we assumed that the instants $\{t_k\}$ to be a stable sampling set, there exists one and only one fixed point and the iteration method will converge to this unique point, which is the original signal. Note that the above argument is also true about any type of interpolations such as an nth order hold polynomial. For the analysis of sample-and-hold and linear interpolation of uniform samples see [9].

IV. SIMULATION RESULTS

To test the proposed method, we have considered a low-pass signal band limited to $W = 100$ Hz. The original signal and the ideal nonuniform samples, at the Nyquist rate, are shown in Fig. 4. The nonuniform samples are initially taken at the Nyquist rate. The instances are chosen randomly such that $|t_k - kT| < (T/4)$; this is a sufficient condition to ensure a stable sampling set. The reconstructed signal from the ideal nonuniform samples at the Nyquist rate is shown in Fig. 5(a) after 10 iterations. The mean-square error (MSE) for the first 10 iterations are shown in Table I under the column "Ideal." If we relax the sufficient condition $|t_k - kT| < (T/4)$, there is no guarantee that the sampling set converges at the Nyquist rate. For a specific sampling set that is to within $|t_k - kT| < (T/2)$, the iterative technique slowly converges; Fig. 5(b) shows the result after 10 iterations.[4] When the samples are totally random, we observe an even slower convergence as shown in Fig. 5(c) and Table II.[5]

At the rates lower than the Nyquist rate, the iterative method diverges. Obviously, if the average sampling rate is higher than

[4]Figs. 5(a)–(g) are optimized for λ experimentally; the optimum values are somewhere between 0.5 and 1.

[5]Clearly, the results for random sampling are different for each realization of the random samples. At the Nyquist rate, they may or may not converge. Even if they converge, the convergence rate depends on the configuration of the random samples. If the samples are clustered in a small interval, the convergence is slow and the segmental MSE is high at other regions of the waveform where there are no samples.

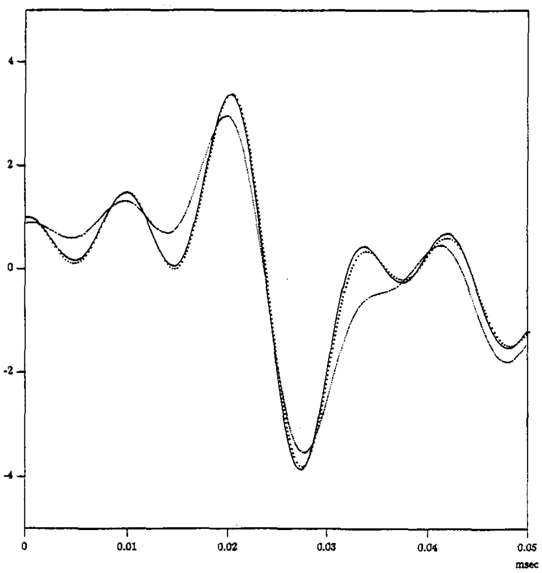

Fig. 7. Reconstruction from sample and hold with variable width at the Nyquist rate ($|t_n - nT| < T/4$). Original signal: · · · · · · · · . Filtered samples: ·········· . After 5 iterations: ——.

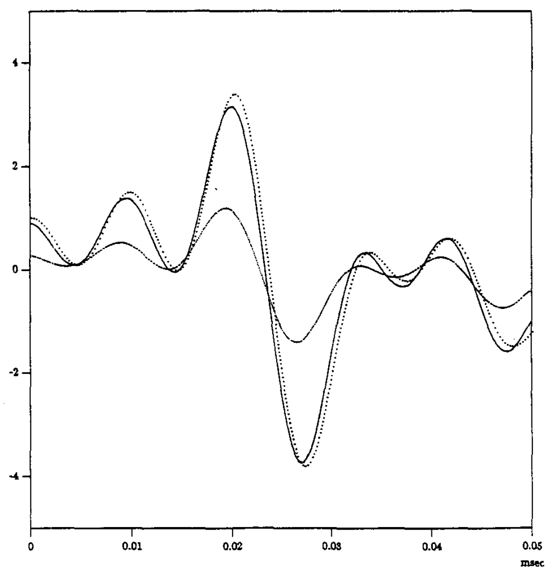

Fig. 8. Reconstruction from natural samples at the Nyquist rate ($|t_n - nT| < T/4$). Original signal: · · · · · · · · . Filtered samples: ·········· . After 5 iterations: ——.

the Nyquist rate, the convergence is guaranteed[6] and is faster. For instance, at twice the Nyquist rate from Fig. 5(d) and Table III we can deduce that if the nonuniform samples are restricted to within $T/4$ where T is the Nyquist interval, we have a faster convergence after 10 iterations compared to the Nyquist sampling. This conclusion is also true when we sample randomly at twice the Nyquist rate, see Fig. 5(e) and Table II. We can get better results if we sample at three times the Nyquist rate as shown in Figs. 5(f) and (g) and Tables II and III.

[6]Random samples at the higher than Nyquist rate is a sampling set and uniquely determine the signal.

For a comparison of the reconstruction from different sampling schemes, we sample at the Nyquist rate and then use the iterative method. The reconstructed signal from the nonuniform samples with zero-order hold (constant and variable width) and the reconstructed signal from natural samples after low-pass filtering (no iteration) and after 5 iterations (we have assumed that $\lambda = 1$ in all cases as discussed in Section III-A) are shown in Figs. 6 through 8, respectively. The pulsewidth in Figs. 2, 6, and 8 is $T/2 = 1/4W$.[7] A comparison of the iteration method for the reconstruction of different kinds of nonuniform sampling schemes (when $|t_k - kT| < (T/4)$, where T is the Nyquist interval) based on MSE is given in Table I.[8] This table shows that for the cases of ideal samples, the iteration converges slightly faster than other sampling schemes.

V. Conclusion of the Experimental Results

We have described a recursive method to recover a band-limited signal from its nonuniform samples. The simulation results show that this method works effectively and fairly fast and the errors after a few iterations are negligible if the sufficient condition $|t_k - kT| < (T/4)$ is satisfied or the sampling rate is higher than the Nyquist rate. We have also shown that this method is also applicable to the other interpolated sampling methods as well. These results and comparisons are summarized in Tables I–III.

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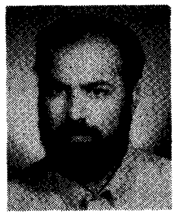
[7]The dependence of MSE on pulsewidth is not significant after 5 iterations.

[8]The oscillations shown in this table for nonideal sampling cases are due to quantization error and approximation of a convolution integral with a discrete one.



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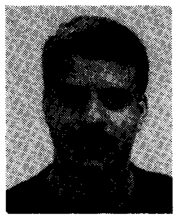

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