Optimum and Suboptimum Memoryless Nonlinearities for the Detection of Ultrashort Light Pulses in Gaussian Noise

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***Abstract*—Detection of ultrashort light pulses with conventional bandlimited photodetectors can lead to substantial performance degradation in digital lightwave communication systems. All-optical nonlinear processing of the received field prior to the optical to electrical conversion can greatly overcome this problem. In this letter, through a decision-theoretic framework and using a basic invariance property of the photodetector, we obtain the optimal zero-memory nonlinear element and evaluate its performance. Possible suboptimum nonlinearities are also investigated and their performance is examined. The results in this letter strongly justify the intuitive use of power-law devices, such as those implemented via two photon absorption (TPA) or second harmonic generation (SHG), in the detection of ultrashort light pulses and show how by using them almost optimal performance can be achieved.**

***Index Terms*—Ultrashort light pulses, nonlinear processing, invariant detectors.**

I. INTRODUCTION

ULTRASHORT light pulses have a prominent role in the future lightwave transmission technology. High-speed time-division multiple-access (TDMA), optical code-division multiple-access (OCDMA), and soliton-based optical communication systems all employ ultrashort light pulses [1]–[3], and even they are also to take part in free space optical (FSO) communication links [4].

Like many other communication systems, additive Gaussian noise is one of the main limiting factors of the performance of many optical communications systems, especially when the dispersion effects and intersymbol interference are negligible, due to the use of low duty-cycle ultrashort light pulses. The origin of the noise can be either the amplified spontaneous emission (ASE) induced by in-line amplifiers, multi-access interference due to spectrally-encoded/time-spreading CDMA, or background radiation in FSO. Optimal detection of such ultrashort light pulses, however, requires ultra-broadband electrooptic and electronic components, and the use of photoreceivers bandlimited to the bit rate could result in substantial performance degradation, and even in some cases can render the performance unacceptable [5],[6].

Nonlinear processing of the received optical field prior to the photodetection is one of the most promising techniques adopted to make the use of common photodetectors a viable solution in such scenarios [7]–[10]. The addition of a proper (all-optical) nonlinear element before the (bandlimited) photodetector yields a better distinction of the signal and noise after the optical to electrical conversion and improves the performance significantly.

All of the proposed nonlinearities so far were on an intuitive basis and no intrinsic optimality is associated with them. In this paper we address the optimum choice of the memoryless nonlinear element, and show how it can be determined via simple decision-theoretic arguments.

In section II, a description of the problem and the system model is given. Section III investigates the optimum and suboptimum receiver structures, and in section IV their performance is evaluated. Section V concludes the paper.

II. SYSTEM MODEL

We consider a conventional intensity-modulation/direct-detection (IM/DD) system. The lowpass equivalent of the received optical field can be written as

$$r(t) = \sum_k a_k p(t - kT_b) e^{j\phi} + n(t) \tag{1}$$

$p(t)$ is the ultrashort light pulse with a duration T_p much smaller than the bit period T_b, ϕ is the phase of the received field which is assumed to be unknown, and $n(t)$ is a complex white Gaussian noise with a two-sided power spectral density N_0. The binary data a_k is assumed to be equiprobable 0 and 1 (OOK modulation). Therefore, if no intersymbol interference is present, decision about each bit is independent of the other bits and we can confine our attention to the following hypothesis test

$$\begin{array}{l} H_0\colon\ r(t) = n(t) \\ H_1\colon\ r(t) = p(t)e^{j\phi} + n(t) \end{array} \quad 0 \leqslant t \leqslant T_b \tag{2}$$

In applications of interest in this paper, the background noise level is sufficiently high for the receiver to be far from a thermal noise or shot noise-limited condition, and the photodetector (with subsequent post-detection electrical filters) can be modeled as a simple energy detector [11],[12].

In systems employing ultrashort light pulses, the bandwidth of the received signal is much larger than the bandwidth of common photodetectors, i.e., the pulse width is much smaller than the integration time of the photodetector, e.g, T, and hence if the noise or interference energy in an interval of length T is comparable to the desired signal energy, it is intuitively evident that the receiver's performance would be totally unacceptable (cf. Section IV).

A proposed – and widely experimented – solution which to some extent can overcome this problem is to perform a

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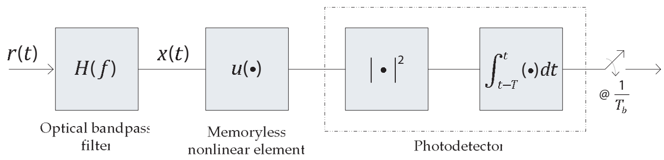


Fig. 1. The general form of the optical nonlinear receiver.

proper all-optical nonlinear processing on the received optical field, prior to passing it to the photodetector, to enhance the detectability of the received pulse. Assuming a zero-memory nonlinear element $u(\cdot)$, the receiver structure in this case takes the general form of Fig. 1, where $H(f)$ is an optical bandpass filter to eliminate the out-of-band noise and produce the signal $x(t) = r(t) * h(t)$, and decision about the received data is based on the samples taken at the bit rate at the output. The main element used so far in the experiments is the optical thresholder (hard-limiter), which by passing the high-amplitude ultrashort light pulse and blocking the low-amplitude noise or interference can improve the receiver performance [14]–[16]. The other elements whose use can boost the receiver performance are the power-law devices (i.e., $u(x) = |x|^\nu$), as they can increase the contrast between signal and noise [17]–[19]. The above choices for the nonlinear element, however, have been picked quite intuitively and the question arises as to how the nonlinear element should be chosen to yield the optimum bit error rate (BER) result.

Due to the very complex (and unknown) dependence of the photodetector's output statistics on the nonlinear element transfer function, this problem cannot be dealt with directly, that is, writing the probability of error as a function of $u(\cdot)$ and then carrying an optimization over it. Next section provides an indirect approach to this problem.

III. Optimum and Suboptimum Nonlinear Elements

In this section the hypotheses test (2) is solved by considering the constraint imposed by the photodetector's finite bandwidth and with an application of the theory of statistical invariance, and we shall see that the structure of the resulting detector will give the optimal nonlinearity immediately. A description of the use of invariance in hypothesis testing and signal detection applications is provided in [20] and [21], respectively.

With the common assumption that the optical receiver is band-limited to the bit rate ($T_b = T$), we consider the hypothesis test (2) with the presumption that the receiver should have the structure shown in Fig. 1. This implies that the decision variable would have the form $\int_0^T |u(x(t))|^2\, dt$. Since $u(\cdot)$ is memoryless and time-invariant, the decision variable remains invariant with a cyclic shift of the signal in the interval $(0, T)$, i.e., $\int_0^T |u(x_T(t+\tau))|^2\, dt$ will be unvaried when τ changes in $(0, T)$, with $x_T(t+\tau) \triangleq x((t+\tau) \bmod T)$. So, should one use a decision variable of this type, and noting that $H(f)$ is time-invariant, the hypothesis test (2) will be invariant under the group of transformations $G_1 = \{g_\tau : g_\tau r(t) = r_T(t+\tau), \tau \in [0, T]\}$.

Moreover, since the phase ϕ is unknown it is easily verified that any rotation in the complex plane also leaves the problem invariant [25]. That is, the problem is invariant under the group $G_2 = \{g_\theta : g_\theta r(t) = e^{j\theta} r(t), \theta \in [0, 2\pi]\}$, and as a result the problem is also invariant under the group $G = G_1 \times G_2$, i.e., the direct product of G_1 and G_2 [23],[25]. Hence, due to the principle of statistical invariance, any reasonable detector should also remain invariant under G, and so our problem is reduced to finding the optimum detector in the class of all detectors which are invariant under this group [20]–[24].

To this end, consider the vector model of the hypothesis test (2), in which after dispensing with the irrelevant noise components, the waveforms $r(t)$, $p(t)$, and $n(t)$ are represented by the complex vectors $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{n}$, each of a finite dimensionality $2BT+1$, with B being the bandwidth of the pulse $p(t)$ [26]. The counterpart of group G for this vector model is the group $\tilde{G} = \{\tilde{g}_{\theta,\tau} : \tilde{g}_{\theta,\tau}\mathbf{r} = e^{j\theta}\mathbf{r}_\tau, \theta \in [0, 2\pi], \tau \in [0, T]\}$, where $\mathbf{r}_\tau$ denotes the vector representation of $r_T(t+\tau)$.

The classical method of obtaining the best invariant detector for an invariant decision problem is to first find the maximal invariant statistics $y = M(\mathbf{r})$ (i.e., an invariant statistics as a function of which any other invariant statistics can be expressed), find its densities under hypotheses H_0 and H_1, and construct the likelihood ratio $L(y)$. That is, the decision rule which yields the minimum probability of error (using the maximum-likelihood (ML) criterion for binary equiprobable data) is given by

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} \underset{H_1}{\overset{H_0}{\lessgtr}} 1 \tag{3}$$

with $f(y|H_i)$ the density of y under hypothesis H_i [20]–[24].

The ratio of the maximal invariant densities in (3) may also be obtained by integrating the densities of $\tilde{g}_{\theta,\tau}\mathbf{r}$ over all $\tilde{g}_{\theta,\tau} \in \tilde{G}$, i.e.,

$$L(y) = \frac{\int_{\tilde{G}} f(\tilde{g}_{\theta,\tau}\mathbf{r}|H_1)\, d\nu(\tilde{g}_{\theta,\tau})}{\int_{\tilde{G}} f(\tilde{g}_{\theta,\tau}\mathbf{r}|H_0)\, d\nu(\tilde{g}_{\theta,\tau})} \tag{4}$$

in which ν is a left invariant Haar measure on $\tilde{G}$ [23],[27],[28]. Note that finding $L(y)$ via the relation (4) does not require the explicit identification of the maximal invariant statistics and its densities, and in some cases (as the case we deal with here) may be more favourable.

It can be shown that for our case $d\nu(\tilde{g}_{\theta,\tau})$ is equal to the differential $d\theta d\tau$ [23], whereupon, noting that $f(\mathbf{r}|H_1) = (2\pi N_0)^{-(2BT+1)} \exp(-\frac{1}{2N_0}\|\mathbf{r} - e^{j\phi}\mathbf{p}\|^2)$, we have from (4)

$$\begin{aligned} L(y) &= \frac{\int_0^T \int_0^{2\pi} \exp(-\frac{1}{2N_0}\|\mathbf{r}_\tau - e^{j(\phi-\theta)}\mathbf{p}\|^2)\, d\theta d\tau}{\int_0^T \int_0^{2\pi} \exp(-\frac{1}{2N_0}\|\mathbf{r}_\tau\|^2)\, d\theta d\tau} \\ &= \frac{\exp(-\frac{E_0}{N_0})}{T} \int_0^T I_0\Big(\Big|\frac{1}{N_0}\int_0^T r_T(t+\tau)p^*(t)dt\Big|\Big) d\tau \end{aligned} \tag{5}$$

where $E_0 = \frac{1}{2}E_p = \frac{1}{2}\int |p(t)|^2 dt$ and $I_0(\cdot)$ is the modified Bessel function of the first kind of order zero. If $p(t)$ is nonzero only in the interval $(0, T_p)$, with $T_p \ll T$, it can be easily verified that $r_T(t+\tau)$ in the above equation can be safely replaced by $r(t+\tau)$, and from (3) we arrive at the decision rule

$$\int_0^T I_0\Big(\Big|\frac{1}{N_0}\int_0^T r(t+\tau)p^*(t)dt\Big|\Big) d\tau \underset{H_1}{\overset{H_0}{\lessgtr}} T e^{\frac{E_0}{N_0}} \tag{6}$$

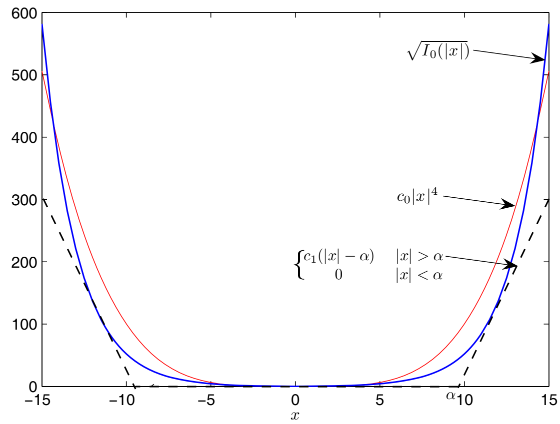


Fig. 2. The transfer function of the optimum and two suboptimum nonlinearities.

Since this detector is the optimum in the class of all G-invariant detectors, it can directly be inferred that the optimal nonlinear element in the receiver structure of Fig. 1 should be chosen to be of the form $u_{opt}(x) = \sqrt{I_0(|x|/N_0)}$. Furthermore, it follows from (6) that the bandpass filter preceding the nonlinearity should be matched to the pulse shape, as might have been expected. However, even though the use of such a nonlinearity yields the best performance, its all-optical implementation may prove to be impracticable. Consequently, some suboptimum but feasible nonlinearities should be sought. From a plot of the transfer function of $u_{opt}(x)$, it can be observed that the aforementioned thresholder and power-law devices may be considered as good approximations to the optimum nonlinear element (Fig. 2).

Theoretically, optical power-law devices can be implemented by phenomena such as νth harmonic generation or ν-photon absorption [29]. Thus far, however, only second harmonic generation (SHG) and two-photon absorption (TPA) have the sufficient conversion efficiency to be used in experiments [17]. Using a combination of SHG and TPA a power-law device with $\nu = 4$ might also be achieved. The ideal optical thresholder, on the other hand, has not been realized yet. Nonetheless, several approximations to the device have been developed using nonlinear optical loop mirror (NOLM) or nonlinear effects in optical fiber, such as self phase modulation (SPM) [7],[14]–[16]. However, almost all of them should be modeled as nonlinear devices with memory, and performance analysis of receivers incorporating them should allow for the exact model of the device. So, in the following we will not consider this kind of nonlinearity.

IV. Performance Analysis of the Receiver

In this section we investigate the performance of the optimum and suboptimum receiver structures obtained above. In addition, it would also be helpful to include the (already known) receiver performance in the two special cases of

1) ideal detection, where the bandwidth of the photodetector is much larger than the signal bandwidth ($T \to 0$), and the receiver reduces to an envelope detector [2],[13], and

2) when no nonlinearity is placed before the photodetector [12].

The latter reveals how much a nonlinearity would improve the performance, and what the result would be were it not to be used. On the other hand, the former provides us with a lower bound to the performance of all detection schemes and can serve as a benchmark to have a correct (optimistic) anticipation of the performance of the receivers to be designed.

We now proceed to the optimum nonlinear detection scheme.

A. The Optimum Nonlinear Receiver

The structure of the optimum nonlinear receiver is given by relation (6). While owing to the known structure of the receiver its performance may be found through Monte Carlo simulation, having a proper approximation of the probability density function (pdf) of the decision statistic would be of great use in the performance analysis, especially in high SNR (very small BER) scenarios.

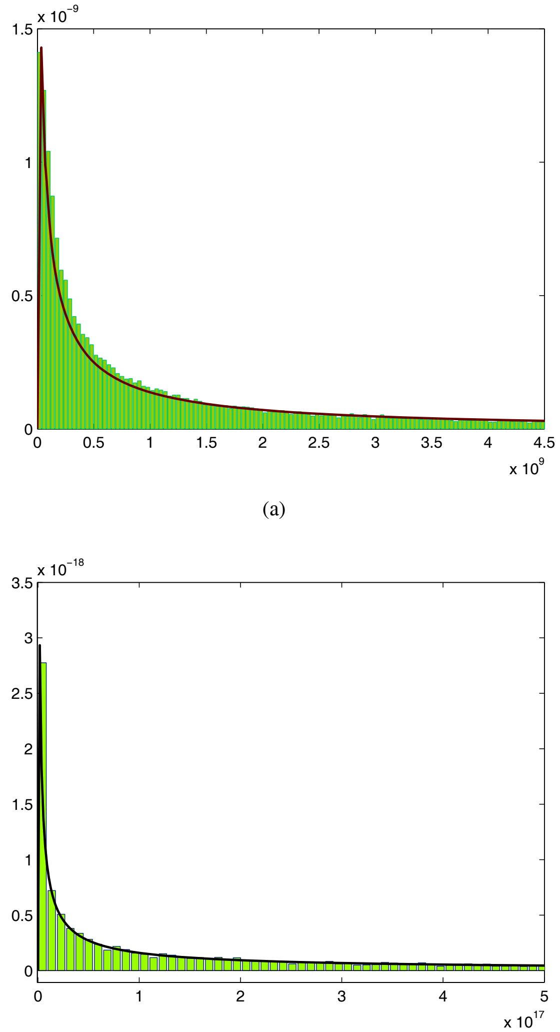
By fitting the simulated histogram of the statistic with a large number of available pdfs, it was concluded that the lognormal distribution would be an accurate approximation in different SNRs (Fig. 3). This result may be intuitively justified noting the exponential behavior of the function $I_0(\cdot)$.

With the standard lognormal density of the form $f(y) = (\sqrt{2\pi}\sigma y)^{-1} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}$, it suffices to find the first and the second moments of y under H_0 and H_1, whence the parameters of the distribution are easily determined. These moments can be obtained by invoking the Taylor series expansion of $I_0(\cdot)$ and using the results of [35] for the (joint) moments of the amplitude of Gaussian processes. Here the results are omitted due to space limitations. Fig. 4 shows the performance of the receiver with the optimum nonlinearity in the cases $T/T_p = 10$ and 100. The conventional Gaussian pulse shape is assumed and the average signal-to-noise ratio (SNR) per bit (for OOK modulation) is $\text{SNR}_b = E_0/2N_0$. Both the simulation results and the BER determined analytically using the lognormal approximation, along with the performance of the ideal detector and the detector with no nonlinearity are presented.

There exists a good agreement between the approximation and simulation results. It is also seen that as expected, the performance of the receiver with no nonlinear element is very poor and a remarkable improvement in the performance comes about with the introduction of the optimal nonlinear element, especially for large values of T/T_p. Note that by using the optimal nonlinearity in a receiver with a bandwidth of about one-tenth of that of the signal, the performance degradation will be less than an order of magnitude, compared to the ideal detection scheme.

B. Suboptimum Nonlinear Receivers

We now examine the performance of receivers incorporating ν-law devices for $\nu = 2$ and 4. The probability distribution of functionals of the form $y = \int_0^T |x(t)|^{2\nu} dt$, with $x(t)$ a complex Gaussian process, have been found only for $\nu = 1$ [30]. When T/T_p is large, by the central limit theorem the normal distribution can be used as a first try. However, a better approximation is to invoke the gamma limit theorem (GLT)



(b)

Fig. 3. Lognormal fit to the histogram of the simulated data. (a) Noise alone, and (b) signal plus noise, at $E_0/N_0 = 14$ dB and $T/T_p = 100$.

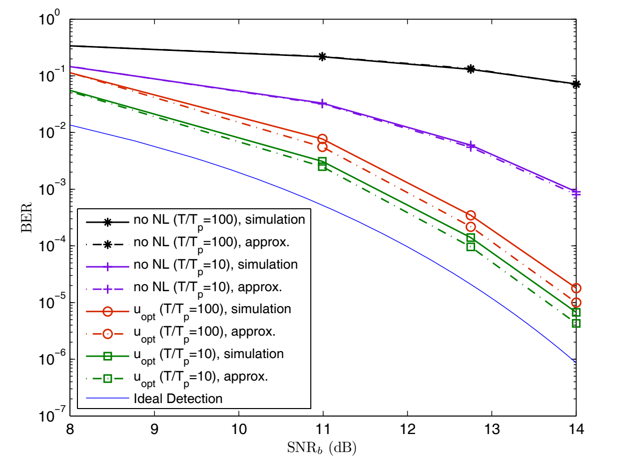


Fig. 4. Performance of the optimum detector, detector with no nonlinearity (NL), and ideal detector.

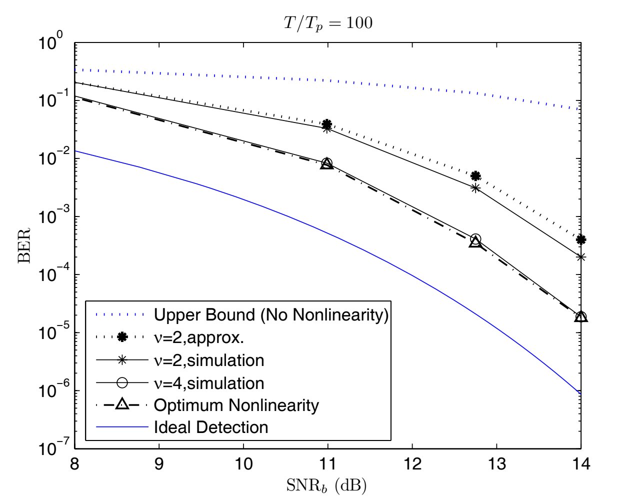


Fig. 5. Performance of power-law devices ($T/T_p = 100$).

which roughly states that a summation of positively skewed random variables converges more rapidly to the gamma density than to normal [31],[32]. While in the case $\nu = 1$ this approximation is very accurate, deviations from gamma are more pronounced when ν increases.

Noting that the gamma pdf is the first term in the Laguerre series expansion of a distribution, the approximation can be improved by considering more terms of the series [33],[34]. Proceeding up to the third term, the modified pdf can be written as $f(y) = e^{-y/\beta}(\frac{y}{\beta})^{\alpha}\left[c_0 + c_3 L_3^{\alpha}\left(y/\beta\right)\right]$, where L_i^{α} is the generalized Laguerre polynomial and α, β, c_0, and c_3 are determined using the first three moments of the desired random variable [35].

Figures 5 and 6 show the performance of power-law receivers for $T/T_p = 100$ and 10, respectively. While the above approximation is fairly accurate for $\nu = 2$, for $\nu = 4$ it underestimates the performance about an order of magnitude and more terms of the Laguerre series should be included to achieve sufficient accuracy. So for $\nu = 4$ only simulation results are presented in these figures.

We notice that the performance improves substantially when we move from $\nu = 1$ (no nonlinearity present) to $\nu = 2$, and as we progress to $\nu = 4$ the improvement is more pronounced for larger values of T/T_p. Most importantly, however, we observe that the performance for $\nu = 4$ is almost identical to that of the optimum nonlinearity and the suboptimality is indeed intangible. Therefore, using (practicable) power-law devices the optimum performance could be achieved. In particular, when T/T_p is not too large (Fig. 6), near optimal detection can be accomplished only by a simple square-law nonlinearity, such as those implemented by TPA or SHG.

V. Conclusion

The problem of detection of ultrashort light pulses, when solved by invoking an invariance property imposed by the photodetector's finite bandwidth, yields the optimum zero-memory nonlinear element that should be used in the receiver structure. Though the use of this element may not be feasible, we showed that using practicable power-law devices significant – or even almost optimal – performance improvement can be accomplished. Particularly, even if the bandwidth of the receiver is about 0.01 of the bandwidth of the ultrashort

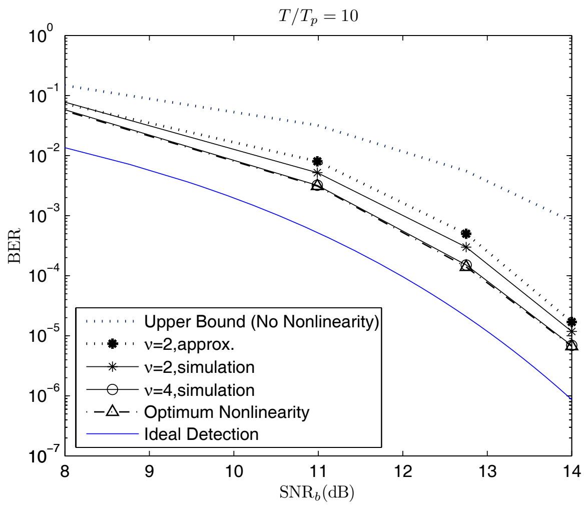


Fig. 6. Performance of power-law devices ($T/T_p = 10$).

light pulse, with less than 2 dB of SNR penalty the ideal matched filter performance could be achieved.

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