Novel Probabilistic Bounds on Power Level Profile of Spectrally-Encoded Spread-Time CDMA Signals

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***Abstract*—In this letter we introduce an effective tool to demonstrate signal power behavior of a typical spectrally-encoded spread-time (SE/ST) CDMA system using probabilistic approach. By defining *power level profile* (*PLP*), as the probability of stretching instantaneous power from a pre-specified threshold, we can find novel upper bounds on the signal power profile. We discuss various properties owing to *PLP* and elaborate on the main parameters affecting the *PLP* value. We present three theorems, propose some simple upper bounds on the *PLP*, and give an insight on the main contributions of the results. We introduce a fundamental parameter, namely β, as a key design parameter which relates the most important quantities affecting the *PLP* behavior. Finally, we propose a corollary demonstrating an approach to guarantee that the *PLP* value is set at zero. It is shown that using suitable distributions, we can also obtain some *distribution gain*, thereby enhancing the overall system performance.**

***Index Terms*—*PLP*, SE/ST CDMA, code design.**

I. Introduction

THE ever increasing demand and ubiquitous use of various wireless communication services alongside the limited availability and highly prized and badly needed radio spectrum have been a major motivation and thrust behind intense research in spectrally efficient modulation techniques. Among many signal design problem in wireless communications, CDMA signals may require the most stringent of all the requirements. CDMA communication systems enjoy from many superior features in comparison with other multiple-access systems, however they may easily suffer from low spectral efficiency if their corresponding signals are not well shaped and designed. Among various CDMA techniques, SE/ST technique is among the most robust and most spectrally efficient techniques [1-5]. In SE/ST CDMA the signature sequence associated to each user is directly applied in the spectral domain of the transmitted pulse as opposed to the time domain in direct sequence CDMA systems. Using spectral domain, pulse shaping can be applied easily by proper manipulation of the amplitudes and phases of various spectral components of the corresponding SE/ST signals which gives the advantages of matching the transmitted spectrum with the channel characteristic.

There are essentially two main reasons in shaping the power profile of an encoded signal in SE/ST systems. First and foremost the success of SE/ST CDMA communication system depends upon the ability to generate a low intensity pseudo-noise signal for each user such that the peak intensity of the encoded signal for other users falls below a certain threshold [6-14]. Secondly, in order to overcome the nonlinear clipping effect of pre- and post-amplifications we need to assure that at all time the maximum value of the encoded signal does not reach above the dynamic range of the linear amplifiers in use [6-19]. For the aforementioned reasons, in this letter we present novel probabilistic bounds to test the peak value of the encoded signal for various classes of phase distributions. The novelty of the bounds are due to its simplicity and it is shown to be as a function of the length of the spectral code, pulse's initial peak power, and a certain threshold and no further optimization is required.

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II. Power Level Profile: Signal Design Model

In a typical SE/ST CDMA system each user employs a signature sequence to encode/decode its corresponding data bits. Each signature sequence is selected among different realization of $\{c_n = \exp(j\varphi_n)\}_{n=-N}^{N}$ in which $\{\varphi_n\}_{n=-N}^{N}$ is an independent and identically distributed random phase sequence with continuous or discrete distribution. A signature code is constructed as [1]

$$c(t) = \frac{\sqrt{P_0}}{2N+1}\mathrm{sinc}(\Omega t)\sum_{n=-N}^{N} c_n \exp(j2\pi n\Omega t) \quad (1)$$

in which P_0 is the peak power, $\Omega = \frac{W}{2N+1}$ is the chip bandwidth, W is the total bandwidth, t represents the continuous time domain and $\mathrm{sinc}(\Omega t) = \sin(\pi\Omega t)/(\pi\Omega t)$. During modulation, $c(t)$ is multiplied by the data stream and then transmitted through a multiple-access channel. At the receiver end a decoder matched to the desired signal, extracts the transmitted data stream. While the desired signal's power is peaked at the output of its corresponding decoder, an interfering user has fluctuating low power signal level. In this scheme the instantaneous power at the output of the encoder has the form of

$$P_{Encoded}(t) = |c(t)|^2 = \frac{P_0}{(2N+1)^2}\mathrm{sinc}^2(\Omega t)\times \sum_{n,n'=-N}^{N} c_n c_{n'}^* \exp(j2\pi(n-n')\Omega t) \quad (2)$$

However, at the output of the decoder we encounter two different cases. The first case, the instantaneous power of the

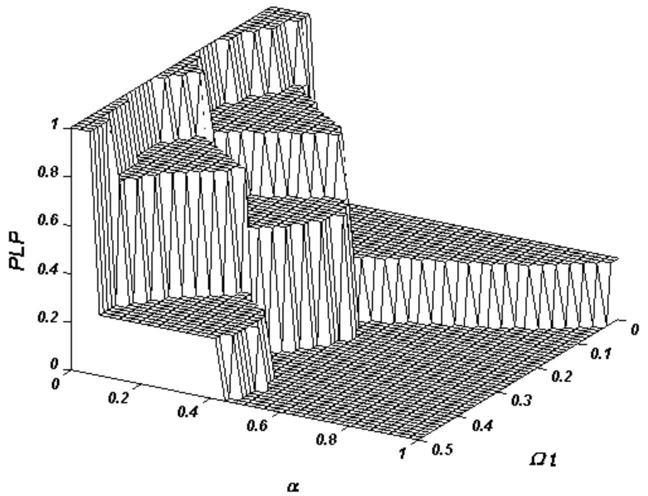

Fig. 1. $PLP(\alpha,t)$ versus α and Ωt for $N = 1$ and $\theta_n \sim \text{uniform}\{k\pi\}_{k=0}^{1}$.

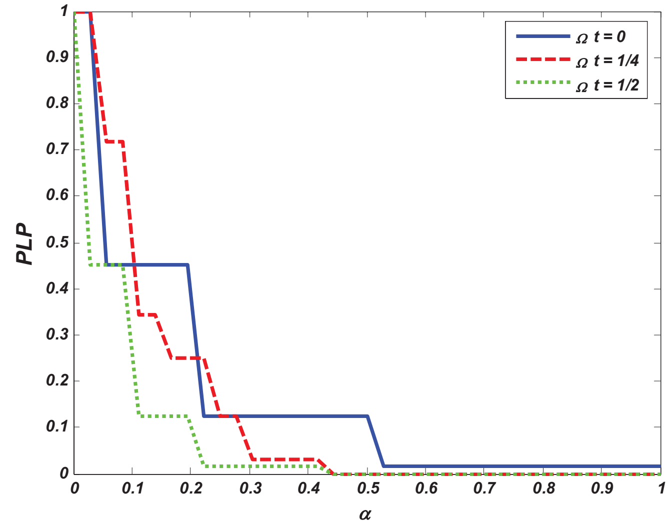

Fig. 2. $PLP(\alpha,t)$ versus α for $N = 3$, $\theta_n \sim \text{uniform}\{k\pi\}_{k=0}^{1}$ and $\Omega t = 0, 1/4, 1/2$.

desired signal has the form of

$$P_{Decoded}^{Desired}(t) = \frac{P_0}{(2N+1)^2}\text{sinc}^2(\Omega t)\times \sum_{n,n'=-N}^{N} \exp(j2\pi(n-n')\Omega t) = P_0\text{sinc}^2(Wt) \tag{3}$$

and the second case, the interfering signal has the form of

$$P_{Decoded}^{Interfering}(t) = \frac{P_0}{(2N+1)^2}\text{sinc}^2(\Omega t)\times \sum_{n,n'=-N}^{N} (c_n c_n'^{*})(c_{n'} c_{n'}'^{*})^{*} \exp(j2\pi(n-n')\Omega t) \tag{4}$$

in which $\{c_n' = \exp(j\varphi_n')\}_{n=-N}^{N}$ denotes the signature sequence of the interfering user. For the sake of notational simplicity, if we define $c_n'' = c_n c_n'^{*} = \exp(j(\varphi_n - \varphi_n'))$ then (4) can be rewritten as

$$P_{Decoded}^{Interfering}(t) = \frac{P_0}{(2N+1)^2}\text{sinc}^2(\Omega t)\times \sum_{n,n'=-N}^{N} c_n'' c_{n'}''^{*} \exp(j2\pi(n-n')\Omega t) \tag{5}$$

$P_{Decoded}^{Interfering}(t)$ in (5) has the same structure as $P_{Encoded}(t)$ in (2) and therefore can be handled in a similar manner. Indeed, if we define θ_n as $\theta_n = \varphi_n$ in $P_{Encoded}(t)$ and $\theta_n = \varphi_n - \varphi_n'$ in $P_{Decoded}^{Interfering}(t)$ then we can work with these two processes in a completely similar manner.

It is interesting to note that while $P_{Decoded}^{Desired}(t)$ is a deterministic function, $P_{Encoded}(t)$ and $P_{Decoded}^{Interfering}(t)$ are two random processes. The specific power level distribution or the instantaneous power for different codes, extended in the range of 0 to P_0, differs from code to code. Therefore $P_{Encoded}(t)$ is a random process with random power level extension. The power extension of $P_{Encoded}(t)$ and/or $P_{Decoded}^{Interfering}(t)$ can be efficiently measured by defining *power level profile* (*PLP*) for $0 \le \alpha \le 1$ as

$$PLP(\alpha,t) = \Pr(\alpha P_0 \le P(t)) \tag{6}$$

in which $P(t)$ may be denoted either as $P_{Encoded}(t)$ or $P_{Decoded}^{Interfering}(t)$ and $\Pr(x \le X)$ is defined as the probability that the random variable X equals or exceeds the real number x [20]. The above definition helps us to investigate important features of the random processes $P_{Encoded}(t)$ and $P_{Decoded}^{Interfering}(t)$. For example, in a real scenario, we may limit the signal power due to the modulator amplifiers or channel distortion effects, or in some cases we intend to decrease the instantaneous power values of the encoder output in order to immune our signals against eavesdropping or to hide the signal below the environmental noise levels. Finally, in a multiple-access environment which is of utmost importance in many wireless communications, at the receiver side we prefer to decrease the power level of the interfering users as much as possible. For all the above examples we require some tools to predict the behavior of the instantaneous power of the encoded signal, i.e., $P_{Encoded}(t)$, or decoded signal, i.e., $P_{Decoded}^{Interfering}(t)$. The *PLP* can be used as an efficient mean to respond to the above inquiries.

In Fig. 1 we show the *PLP* as a function of α and Ωt for the system with $N = 1$ and $\theta_n \sim \text{uniform}\{k\pi\}_{k=0}^{1}$. In this figures we only sketch the *PLP* value for $0 \le \Omega t \le 0.5$. Primarily, this is due to the fact that first, the $\text{sinc}^2(\Omega t)$ term in the definition of $P(t)$ has its main energy in the vicinity of $t = 0$ and second, since the *PLP* function is even with respect to t. We can clearly see the existence of different power levels for different values of α and Ωt. Although for this figure we have chosen a very simple example but yet the *PLP* is a complicated function of α and Ωt.

The *PLP* value, in general, depends on four main factors, namely α, Ωt, N and θ_n distribution. α is the power threshold indicating a level for testing $P(t)$ values, Ωt is the *PLP* time dependence parameter, and the values of N and θ_n's probability distribution are two factors indicating the effects

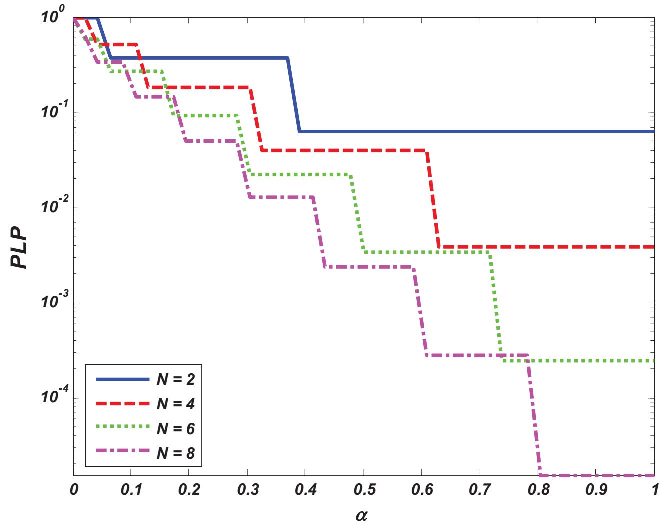

Fig. 3. $PLP(\alpha, t)$ versus α for $\Omega t = 0$, $\theta_n \sim \text{uniform}\{k\pi\}_{k=0}^{1}$ and $N = 2, 4, 6, 8$.

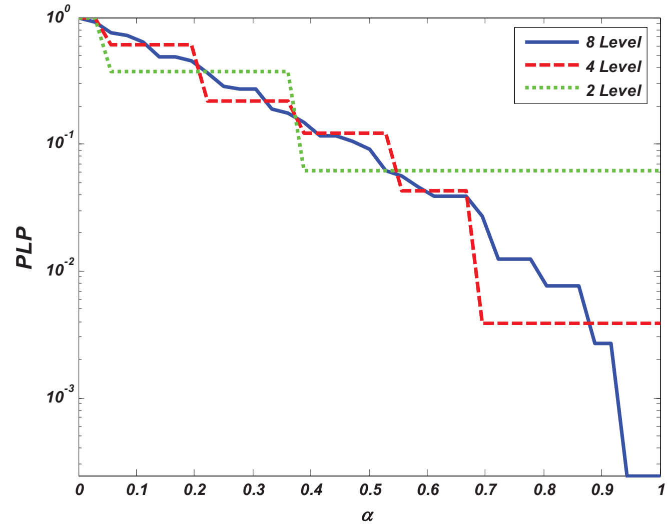

Fig. 4. $PLP(\alpha, t)$ versus α for $N = 2$, $\Omega t = 0$ and $\theta_n \sim$ uniform with 2, 4 or 8 levels.

of $\{\theta_n\}_{n=-N}^{N}$ phase sequence. In general, there is a complex relation among the above parameters and the *PLP* values. Therefore, in most cases we are more interested in finding some upper bounds on the *PLP* values as a simple function of the above parameters.

In Fig. 2, we depict $PLP(\alpha, t)$ versus α for $N = 3$ and $\Omega t = 0, 1/4, 1/2$, with equal weight binary distribution for θ_n, i.e., $\theta_n \sim \text{uniform}\{k\pi\}_{k=0}^{1}$. It can be seen that *PLP* values are decreasing function of α. Moreover, there is an important finding: There is not any t_0 such that $PLP(\alpha, t) \leq PLP(\alpha, t_0)$ for all values of α. Indeed, based on Fig. 2, for some values of α, $PLP(\alpha, t) \leq PLP(\alpha, 0)$ and for some others $PLP(\alpha, t) \leq PLP(\alpha, 1/4\Omega)$. This phenomenological behavior indicates a challenge on finding simple upper bounds, as a function of α, for $PLP(\alpha, t)$ based on time domain search.

In Fig. 3, we depict $PLP(\alpha, t)$ versus α for $\Omega t = 0$ and $\theta_n \sim \text{uniform}\{k\pi\}_{k=0}^{1}$ with $N = 2, 4, 6, 8$. It can be easily seen that, in most cases, increasing N can effectively decrease the *PLP* values. Of course there are some values of α in which increasing N leads to a higher *PLP* value. The general behavior of *PLP* versus N highlights the important role of this parameter in order to control the *PLP* values.

In Fig. 4, we show $PLP(\alpha, t)$ versus α for $N = 2$ and $\Omega t = 0$ with three different θ_n distributions, namely $\text{uniform}\{k\pi\}_{k=0}^{1}$, $\text{uniform}\{k\pi/2\}_{k=0}^{3}$, and $\text{uniform}\{k\pi/4\}_{k=0}^{7}$. We consider these as 2, 4, and 8 level distributions, respectively. We observe that increasing the distribution level leads to a more smooth *PLP* function. Although, it is clear that *PLP* is a complex function of θ_n distributions, but there is not a tangible differences in the *PLP* values at moderate values of α. But for larger values of α, higher order distributions have the ability to reduce *PLP* values far beyond the lower order counterparts. Therefore, the main effect of θ_n distributions can be seen only at larger values of α.

III. The *PLP* Bounds

In this section we introduce simple upper bounds on the *PLP* function. To this end, we impose some conditions on the distribution of φ_n which in general are not strictly limiting. The results are summarized in the following theorems.

Theorem 1: Let $p(\varphi)$ for $-\pi < \varphi \leq \pi$ denotes φ_n's distribution with the following properties:

a) $p(-\varphi) = p(\varphi)$, i.e., $p(\varphi)$ is an even distribution.

b) $\mathbf{E}\cos(\varphi_n) = 0$ in which $\mathbf{E}$ denotes the expectation operator.

c) $\mathbf{E}\cos(m\varphi_n) \geq 0$ for $m = 2, 3, 4, \ldots$.

Then for all values of t and $m = 0, 1, 2, \ldots$

$$\mathbf{E}P^m(t) \leq \mathbf{E}P^m(0) \tag{7}$$

in which $P(t)$ may be regarded either as $P_{Encoded}(t)$ or $P_{Decoded}^{Interfering}(t)$.

Theorem 2: Under the assumptions of *theorem 1*, $PLP(\alpha, t)$ has an *N*-asymptote bound as

$$PLP(\alpha, t) \leq \frac{(2m)!}{m! \times (2\beta)^m} \quad : \quad m = \lfloor(\beta + 1)/2\rfloor \tag{8}$$

in which $\beta = \alpha(2N + 1)$ and $\lfloor x \rfloor$ denotes the greatest lower integer part of x.

Theorem 3: If as an extreme example, we consider uniform distribution on $[-\pi, \pi]$, then $PLP(\alpha, t)$ has an *N*-asymptote bound as

$$PLP(\alpha, t) \leq \frac{m!}{\beta^m} \quad : \quad m = \lfloor \beta \rfloor \tag{9}$$

For the proofs, see section V.

1- *Theorem 1* proposes a tight bound on the *m*th moment of $P(t)$, i.e., $\mathbf{E}P^m(t)$, based on its equivalent counterpart for $P(0)$. This bound not only is used in proof of *theorem 2*, but also removes the role of t in our derivations.

2- Although in *theorem 1* we can relax t and work only with $t = 0$, but the results are not enough to directly compare the *PLP* values. In other words, from $\mathbf{E}P^m(t) \leq \mathbf{E}P^m(0)$ for $m = 0, 1, 2, \ldots$ we can not directly infer that $PLP(\alpha, t) \leq$

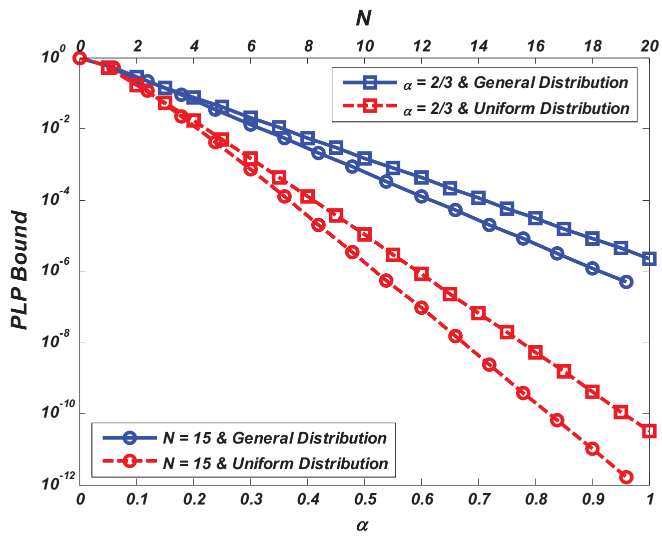

Fig. 5. Bounds of *theorem 2* and *theorem 3* versus α for $N = 15$ and versus N for $\alpha = 2/3$.

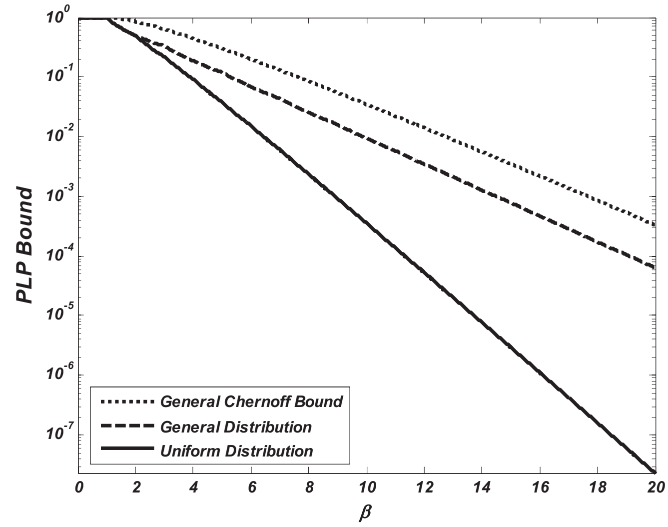

Fig. 6. Bounds of *theorem 2* and *theorem 3* and the Chernoff bound versus *design factor* β. This figure can be considered as a *design curve*.

$PLP(\alpha, 0)$. This fact, which had been shown in previous section following the discussions on Fig. 2 compels us to obtain the upper bounds of (8) and (9).

3- *Theorem 2* proposes a general bound on *PLP* values which can be computed in a simple manner. In fact, the only required parameters we need are the values of α and N in the product form of $\beta = \alpha(2N + 1)$.

4- The bound on *theorem 2* is independent of t, hence it is uniformly applicable for all values of t. This property significantly reduces the complexity associated with the exact value of *PLP* in relation with parameter t.

5- The bound on *theorem 2* is independent of the exact form of φ_n distribution as long as its distribution satisfies the weak conditions of *theorem 1*. Uniform distributions are among the simplest and most extensively used examples. Therefore, the above bound is also uniformly applicable for a large set of φ_n distributions.

6- By *N*-asymptote we imply that the above bounds demonstrate the limiting behavior of *PLP* for large enough values of N. Indeed, the main applications arise mostly for large values of N, and for smaller values the exact behavior of *PLP* function can be investigated by simple computer simulations.

7- The bound introduced in *theorem 2* is as a function of the most important parameter namely $\beta = \alpha(2N + 1)$. The dominant functional behavior of the bound is in the form of β^{-m}, in which m itself increases as β increases. This indeed induces a fast decay in β which can be shown to be faster than exponential bounds such as Chernoff bound.

8- The results of *theorem 3* are a special case of results of *theorem 2* and therefore all the above discussions are also applicable in this case. The only difference is in specifying φ_n distribution which allows us to obtain a special tighter bound.

IV. More Discussion on *PLP* Bound

In this section we intend to expand our scope and discuss in more detail certain applications using the results of the above theorems. We begin our discussion by sketching the bounds of *theorem 2* and *theorem 3* in Fig. 5, in which we plot the *PLP* bounds versus α for $N = 15$. We consider two cases of general distribution, satisfying the conditions of *theorem 1*, and uniform distributions over $[-\pi, \pi]$ separately. It can be observed that using uniform distribution may lead to a great performance improvement, i.e., *PLP* values as small as 10^{-12}. Furthermore and more directly, in this figure we investigate the effects of N on the *PLP* bounds through plotting the bounds versus N for $\alpha = 2/3$ considering two different φ_n distributions. Similarly, using uniform distribution is more preferable than a nonuniform replica, especially for larger values of N.

The above discussions and further examining (8) and (9) lead us to sketch the *PLP* bounds versus $\beta = \alpha(2N + 1)$ in Fig. 6. We also plot the Chernoff bound for comparison. In this figure we can see the combined effects of α and N. It is clear that by controlling the β value one can efficiently control the *PLP* values. For example, consider a system design problem in which one requires to guarantee the performance criterion of $PLP \leq 10^{-3}$ using equal weight binary ± 1 codes. In this case and by referring to Fig. 6, one infers that setting $15 \leq \beta$ would be sufficient. Now, for different values of α, the proper value for N is obtained from $\beta = \alpha(2N + 1)$. For example for $\alpha = 1/2$ we have $N = 15$, and for $\alpha = 1/4$, $N = 30$. Hence, we may consider Fig. 6 as *design curve* and β as the *design factor* of a SE/ST CDMA system. It is clear that we obtain some improvements in performance by resorting to the uniform distribution. This improvement may be considered as *distribution gain*, in two cases of *PLP Gain* versus β or β *Gain* versus *PLP*. In other words, using uniform distribution would lead to a large *PLP Gain*, a large amount of decrease in the *PLP* value, compared with a nonuniform or general counterparts. Similarly, uniform distribution can be used as a tool to save design resources. For example, for *PLP* set as 10^{-6} and by simple evaluation, we can save a factor of approximately 1.7 in selecting N or α values. Another important result which is related to the previous theorems can be presented in the following corollary.

Corollary 1: Assume in a typical SE/ST CDMA system, we insist on setting $\Pr(\alpha P_0 \leq P(t)) = 0$ for a pre-specified value of α. This can be done if we remove at most $\frac{100\times(2m)!}{m!\times(2\beta)^m}$ percent of codewords from our codebook, in which $m = \lfloor(\beta+1)/2\rfloor$ and $\beta = \alpha(2N+1)$. Moreover, by using uniform distribution we can reduce this percentage to $\frac{100\times m!}{\beta^m}$ with $m = \lfloor\beta\rfloor$.

For example in a system with $N = 20$ and $\alpha = 1/5$, we need to remove at most 3% of the codewords from the nonuniform codebook or 0.2% of codewords from the uniform codebook (see Fig. 6). This will result that, we can guarantee $\Pr(P_0/5 \leq P(t)) = 0$ for the remaining new codebook. This example shows the penalty for setting *PLP* exactly at zero is negligible in most practical cases.

V. Proof of Theorems

In this section we outline the proof of theorems in section III.

Lemma 1: if φ_n satisfies the conditions of *theorem 1*, then θ_n satisfies the same conditions, too.

Proof: In the case of $P_{Encoded}(t)$, $\theta_n = \varphi_n$, hence *lemma 1* directly applies. However, in the case of $P_{Decoded}^{Interfering}(t)$ we have $\theta_n = \varphi_n - \varphi'_n$, in which for both independent random variables φ_n and φ'_n satisfy the conditions of *theorem 1*. Here, condition *(a)* is trivial since distribution function of θ_n is obtained by the convolution of two even distribution functions. Moreover, we note that

$$\begin{aligned}\mathbf{E}\cos(m\theta_n) &= \mathbf{E}\cos(m\varphi_n - m\varphi'_n)\\ &= \mathbf{E}\cos(m\varphi_n)\cos(m\varphi'_n)\\ &= \mathbf{E}\cos(m\varphi_n)\times\mathbf{E}\cos(m\varphi'_n)\end{aligned} \tag{10}$$

Therefore, θ_n satisfies conditions *(b)* and *(c)*, following φ_n and φ'_n. Based on *lemma 1*, in the following we only work with θ_n notation.

Proof of *theorem 1*: From (2) or (5) and for $m = 0, 1, 2, \ldots$ we have

$$\begin{aligned}\mathbf{E}P^m(t) = &\frac{P_0^m}{(2N+1)^{2m}}\times\mathrm{sinc}^{2m}(\Omega t)\times\\ &\sum_{n_1,\ldots,n_m=-N}^{N}\ \sum_{n'_1,\ldots,n'_m=-N}^{N}\mathbf{E}(c_{n_1}\cdots c_{n_m}c^*_{n'_1}\cdots c^*_{n'_m})\\ &\times\exp(j2\pi(n_1-n'_1+\cdots+n_m-n'_m)\Omega t)\end{aligned} \tag{11}$$

Since $\{c_n = \exp(j\theta_n)\}_{n=-N}^{N}$ is an i.i.d. random sequence we can write

$$\begin{aligned}\mathbf{E}P^m(t) = &\frac{P_0^m}{(2N+1)^{2m}}\times\mathrm{sinc}^{2m}(\Omega t)\times\\ &\sum_{n_1,\ldots,n_m=-N}^{N}\ \sum_{n'_1,\ldots,n'_m=-N}^{N}\mathbf{E}(c_{-N}^{\gamma_{-N}}\cdots c_N^{\gamma_N})\\ &\times\exp(j2\pi(n_1-n'_1+\cdots+n_m-n'_m)\Omega t)\\ = &\frac{P_0^m}{(2N+1)^{2m}}\times\mathrm{sinc}^{2m}(\Omega t)\times\\ &\sum_{n_1,\ldots,n_m=-N}^{N}\ \sum_{n'_1,\ldots,n'_m=-N}^{N}\mathbf{E}c_{-N}^{\gamma_{-N}}\cdots\mathbf{E}c_N^{\gamma_N}\\ &\times\exp(j2\pi(n_1-n'_1+\cdots+n_m-n'_m)\Omega t)\end{aligned} \tag{12}$$

in which $\{\gamma_n = \gamma_n(n_1,\ldots,n_m,n'_1,\ldots,n'_m)\}_{n=-N}^{N}$ are some integer coefficients. Based on the assumptions of *theorem 1* $\mathbf{E}c_n^m = \mathbf{E}\cos(m\theta_n) \geq 0$ and since $P(t)$ is real we have

$$\begin{aligned}\mathbf{E}P^m(t) = &\frac{P_0^m}{(2N+1)^{2m}}\times\mathrm{sinc}^{2m}(\Omega t)\times\sum_{n_1,\ldots,n_m=-N}^{N}\\ &\sum_{n'_1,\ldots,n'_m=-N}^{N}\mathbf{E}\cos(\gamma_{-N}\theta_{-N})\cdots\mathbf{E}\cos(\gamma_N\theta_N)\\ &\times\cos(2\pi(n_1-n'_1+\cdots+n_m-n'_m)\Omega t)\end{aligned} \tag{13}$$

However, since $\mathbf{E}\cos(\gamma_{-N}\theta_{-N})\cdots\mathbf{E}\cos(\gamma_N\theta_N) \geq 0$ we can write

$$\begin{aligned}\mathbf{E}P^m(t) \leq &\frac{P_0^m}{(2N+1)^{2m}}\times\sum_{n_1,\ldots,n_m=-N}^{N}\\ &\sum_{n'_1,\ldots,n'_m=-N}^{N}\mathbf{E}\cos(\gamma_{-N}\theta_{-N})\cdots\mathbf{E}\cos(\gamma_N\theta_N)\end{aligned} \tag{14}$$

But the right hand side of (14) is equal to the exact value of $\mathbf{E}P^m(0)$ and therefore for all values of t and for $m = 0, 1, 2, \ldots$, $\mathbf{E}P^m(t) \leq \mathbf{E}P^m(0)$, which completes the proof.

Proof of *theorem 2*: Based on (6), we have defined *PLP* as

$$PLP(\alpha, t) = \Pr(\alpha P_0 \leq P(t)) = \int_{\alpha P_0}^{\infty} p(P,t)dP \tag{15}$$

in which $p(P,t)$ denotes probability distribution of $P(t)$. However, since $P(t) \geq 0$, it can be easily seen that for $m = 0, 1, 2, \ldots$

$$\begin{aligned}\int_{\alpha P_0}^{\infty} p(P,t)dP &\leq \int_0^{\infty}(P/\alpha P_0)^m p(P,t)dP\\ &= \mathbf{E}(P(t)/\alpha P_0)^m\end{aligned} \tag{16}$$

Now using the results of *theorem 1*, and combining (15) and (16) we have

$$PLP(\alpha, t) \leq \mathbf{E}P^m(0)/(\alpha P_0)^m \tag{17}$$

The above relation can be further simplified if we focus on $P(0)$. However,

$$P(0) = \frac{1}{2N+1}\times\Big|\sqrt{\frac{P_0}{2N+1}}\sum_{n=-N}^{N}c_n\Big|^2 = \frac{c_R^2 + c_I^2}{2N+1} \tag{18}$$

in which $c_R = \sqrt{\frac{P_0}{2N+1}}\sum_{n=-N}^{N}\cos(\theta_n)$ and $c_I = \sqrt{\frac{P_0}{2N+1}}\sum_{n=-N}^{N}\sin(\theta_n)$ are the summation of i.i.d. random variables. Under the assumptions of *theorem 1* and in the limit when $N \longrightarrow \infty$, c_R and c_I tend to be two independent zero-mean Gaussian random variables with variances of $P_0\mathbf{E}\cos^2(\theta_n)$ and $P_0\mathbf{E}\sin^2(\theta_n)$, respectively. If we define

$\sigma^2 = \mathbf{E}\cos^2(\theta_n)$ and using Gaussian properties we have

$$\begin{aligned}\mathbf{E}P^m(0) &= \frac{1}{(2N+1)^m}\sum_{k=0}^{m}\binom{m}{k}\mathbf{E}c_R^{2k} \times \mathbf{E}c_I^{2(m-k)} \\ &= (\frac{P_0}{2N+1})^m \sum_{k=0}^{m}\binom{m}{k} \times 1 \times 3 \times \cdots \times (2k-1) \\ &\times 1 \times 3 \times \cdots \times (2(m-k)-1) \times \sigma^{2k}(1-\sigma^2)^{m-k} \\ &\leq (\frac{P_0}{2N+1})^m \sum_{k=0}^{m}\binom{m}{k} \times 1 \times 3 \times \cdots \times (2k-1) \\ &\times (2k+1) \times \cdots \times (2m-1) \times \sigma^{2k}(1-\sigma^2)^{m-k} \\ &= (\frac{P_0}{2N+1})^m \times 1 \times 3 \times \cdots \times (2m-1) \times \\ &\sum_{k=0}^{m}\binom{m}{k}\sigma^{2k}(1-\sigma^2)^{m-k} = (\frac{P_0/2}{2N+1})^m \times \frac{(2m)!}{m!}\end{aligned} \tag{19}$$

It should be mentioned that the above bound is equivalent to the exact value of $\mathbf{E}P^m(0)$ for $\sigma^2 = 0$ or 1, which accounts for binary uniform distribution. Furthermore by using (19) in (17) and defining $\beta = \alpha(2N+1)$ we have

$$PLP(\alpha, t) \leq \frac{(2m)!}{m! \times (2\beta)^m} \tag{20}$$

The above relation is valid for all $m = 0, 1, 2, \ldots$ and therefore, we would find the optimum m through minimizing the right hand side of (20). The minimization result can be written as a function of β, i.e.,

$$m = \lfloor \frac{\beta+1}{2} \rfloor \tag{21}$$

(20) together with (21) completes the proof.

Proof of *theorem 3*: If $\theta_n \sim \text{uniform}[-\pi, \pi]$, then $\sigma^2 = 1/2$ and

$$\begin{aligned}\mathbf{E}P^m(0) &= (\frac{P_0/2}{2N+1})^m \sum_{k=0}^{m}\binom{m}{k} \times 1 \times 3 \times \cdots \times (2k-1) \\ &\times 1 \times 3 \times \cdots \times (2(m-k)-1) \\ &= (\frac{P_0/2}{2N+1})^m \times 2^m m! = (\frac{P_0}{2N+1})^m \times m!\end{aligned} \tag{22}$$

The remaining parts of the proof are similar to the above, and the final result can be expressed as

$$PLP(\alpha, t) \leq \frac{m!}{\beta^m} \quad : \quad m = \lfloor \beta \rfloor \tag{23}$$

VI. Conclusion

In this letter we discussed on temporal power profile of a typical SE/ST CDMA signal. For this, we introduced the concept of *PLP* function as a useful tool. The *PLP* function, defined as $PLP(\alpha, t) = \Pr(\alpha P_0 \leq P(t))$, is a probabilistic measure which determines the extension behavior of $P(t)$ as a function of time. We discussed on the properties of *PLP* and demonstrated some simple facts through a few figures. In general it was shown that *PLP* is a function of the power threshold α, time dependency parameter Ωt, and the phase sequence $\{\theta_n\}_{n=-N}^{N}$.

In order to present a deep insight to the general behavior, we introduced 3 theorems. In these theorems, we proposed upper bounds on *PLP* values which have some interesting properties such as, uniform applicability for all values of t and/or for a large class of θ_n distributions, dependence only on the design factor $\beta = \alpha(2N+1)$, and fast decay rate. Finally, we proposed a corollary, suggesting a simple way to set the *PLP* value at exactly zero.

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