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Symmetric relaying based on partial decoding and the capacity of a class of relay networks

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Abstract: Symmetric relaying is a method of relaying in which the relays can decode the message of other relays in the network in addition to the source message. In this paper an achievable rate is presented for a symmetric two-relay network based on partial decoding. The strategy make use of familiar techniques such as product binning, regular encoding/sliding window decoding and regular encoding/backward decoding. The proposed rate is shown to subsume the previously proposed rate for feed-forward relay network based on decode-and-forward. This rate is also used to establish the capacity of a generalisation of Aref network called 'semi-deterministic relay network with no interference at the relays' and independent relay inputs.

1 Introduction

The discrete-memoryless relay network denoted by $(\mathcal{X}_0 \times \mathcal{X}_1 \times \cdots \times \mathcal{X}_N, p(y_0, y_1, \ldots, y_N|x_0, x_1, \ldots, x_N), \mathcal{Y}_0 \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N)$ consists of a sender $X_0 \in \mathcal{X}_0$, a receiver $Y_0 \in \mathcal{Y}_0$, relay senders $X_1 \in \mathcal{X}_1, \ldots, X_N \in \mathcal{X}_N$ and relay receivers $Y_1 \in \mathcal{Y}_1, \ldots, Y_N \in \mathcal{Y}_N$ and a family of conditional probability mass functions $p(y_0, y_1, \ldots, y_N|x_0, x_1, \ldots, x_N)$ on $\mathcal{Y}_0 \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N$ one for each $(x_0, x_1, \ldots, x_N) \in \mathcal{X}_0 \times \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$. A $(2^{nR}, n)$ code for the channel consists of: (i) a set of messages $1, 2, \ldots, 2^{nR}$, (ii) an encoding function that maps each message w into a codeword $x^n(w)$ of length n, (iii) relay encoding functions $x_{1i} = f_i(y_{11}, y_{12}, \ldots, y_{1,i-1})$, for $1 \le i \le n$ and (iv) a decoding function that maps each received sequence y^n into an estimate $\hat{w}(y^n)$. A rate R is achievable if there exists a sequence of $(2^{nR}, n)$ codes with $P_e^{(n)} = P(\hat{W} \neq W) \to 0$, as $n \to \infty$. Channel capacity C is defined as the supremum over the set of achievable rates.

The relay channel, first introduced by Van der Meulen in [1], describes a single-user communication channel where a relay helps a sender–receiver pair in their communication. In [2], Cover and El Gamal proved a converse result for the relay channel, the so-called max-flow min-cut upper bound. Additionally, they established two coding approaches and three achievability results for the discrete-memoryless relay channel. They also presented the capacity of degraded, reversely degraded relay channel and the relay channel with full feedback. In [3], partial decoding scheme or generalised block Markov encoding was defined as a special case of the proposed coding scheme by Cover and El Gamal [2], Theorem 7]. In this encoding scheme, the relay does not completely decode the transmitted message by the sender. Instead, the relay decodes only part of the message transmitted by the sender. Partial decoding scheme was used to establish the capacity of two classes of relay channels called semi-deterministic relay channel [3–4] and orthogonal relay channel [5].

The last few decades have seen tremendous growth in communication networks. The most popular examples are cellular voice, data networks and satellite communication systems. These and other similar applications have motivated researchers to extend Shannon's information theory to networks. In the case of relay networks, deterministic relay networks with no interference, first introduced by Aref [4]. Aref determined the unicast capacity of such networks. The multicast capacity of Aref networks is also characterised in [6].

There are two common protocols for relaying in a network: (i) decode-and-forward and (ii) compress-and-forward proposed in [2], that are extensively used for relaying in the networks [7–13]. In [9] and [10], partial decoding scheme was extended to multi-relay network and the capacity of multilevel semi-deterministic and orthogonal relay network was established. In [11], mixed strategy consisting of partial decoding scheme and compress-and-forward was developed for multiple relay networks. In [12], a comprehensive partial decoding scheme based on regular encoding/sliding window decoding analysis was proposed, in which in contrast to [9–11], all possible partial decoding states are considered between different parts of the messages of the source and the relays in a two-level relay network. In this way, the common and private parts of the message transmitted by the source are defined to be decoded by the appropriate relays. In the proposed relaying methods especially those which are based on decode-and-forward [9–12], it is assumed that the relays are arranged in the feed-forward structure from the source to the destination, i.e. the message transmitted by the ith relay can only be decoded by the jth relay ($j > i$), and cannot be decoded by the previous relays. This is a kind of multi-hopping. In [14] a different method of relaying named symmetric relaying was proposed. In symmetric relaying, the relays are arranged in the same position and no priorities are considered for any of them over the others in receiving the message of the source or the message of the other relays. The concept of symmetric relaying had been implicitly mentioned in Aref's thesis [4] in 1980 to prove the capacity of a deterministic relay network with no interference (Aref network). In [14], the concept of symmetric relaying was explicitly defined for two-relay networks, in which each relay can completely decode the message of the other relay in addition to some part of the source message.

In this work we propose a new achievable rate for a two-relay network based on symmetric relaying and partial decoding. In the proposed rate, each relay can partially decode the message of the other relay and the source message. The proof involves current techniques such as product binning [15], regular encoding/sliding window decoding [7] and regular encoding/backward decoding [16]. Then by using the proposed achievable rate we present the capacity of a generalisation of Aref network named 'semi-deterministic relay network with no interference and independent relay inputs'.

2 Symmetric relaying scheme

In [3], partial decoding is defined as a special case of Theorem 7 given in [2]. In this encoding scheme, the relay does not completely decode the transmitted message by the sender. Instead, the relay decodes only part of the message transmitted by the sender. A block Markov encoding time–frame is used in this scheme such that the relay decodes part of the message transmitted in the previous block and cooperates with the sender to transmit the decoded part of the message to the sink in current block.

In this section, we apply the concept of partial decoding scheme to the relay networks with two relays and present parallel relaying scheme. In Fig. 1, individual parts of the messages are shown. In this figure, (U_{01}^1, U_{01}^2) denotes part of the source message that is decoded by the first relay.

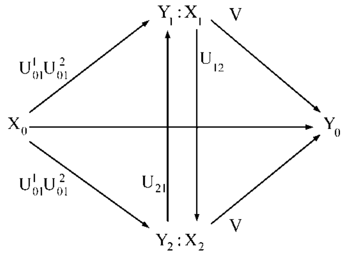

Figure 1 *Relay networks with the individual parts of the messages at the sender and the relays shown in it*

(U_{02}^1, U_{02}^2) denotes part of the source message that is decoded by the second relay. U_{12} denotes part of the transmitted message by the first relay that is decoded by the second relay. U_{21} denotes part of the transmitted message by the second relay that is decoded by the first relay. V denotes the cooperation variable. The rates of variables $U_{01}^1, U_{01}^2, U_{12}, U_{02}^1, U_{02}^2, U_{21}$ are shown by $R_{01}^1, R_{01}^2, R_{12}, R_{02}^1, R_{02}^2, R_{21}$, respectively. The rate of the part of the message source that is directly decoded by the receiver is shown by R_{00}. The proposed rate is shown by the next theorem.

Theorem 1: For any relay networks $(\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2, p(y_0, y_1, y_2|x_0, x_1, x_2), \mathcal{Y}_0 \times \mathcal{Y}_1 \times \mathcal{Y}_2)$, the capacity C is lower bounded by

$$C \geq \sup_{p(v,u_{12},u_{21},u_{01}^1,u_{02}^1,u_{01}^2,u_{02}^2,x_0,x_1,x_2)} \min \left\{ \begin{array}{l} I(X_0X_1X_2; Y_0), \\ I(U_{01}^1U_{01}^2; Y_1|X_1U_{21}U_{12}V) + I(U_{02}^2U_{21}; Y_1|X_1U_{12}V) \\ \quad +I(X_0X_2; Y_0|X_1U_{01}^1U_{01}^2U_{02}^2U_{12}U_{21}V), \\ I(U_{02}^1U_{02}^2; Y_2|X_2U_{21}U_{12}V) + I(U_{01}^2U_{12}; Y_2|X_2U_{21}V) \\ \quad +I(X_0X_1; Y_0|X_2U_{02}^1U_{01}^2U_{02}^2U_{12}U_{21}V), \\ I(U_{02}^2U_{21}; Y_1|X_1U_{12}V) + I(U_{01}^2U_{12}; Y_2|X_2U_{21}V) \\ \quad +I(X_0X_1X_2; Y_0|U_{01}^2U_{02}^2U_{12}U_{21}V), \\ I(U_{01}^1U_{01}^2; Y_1|X_1U_{12}U_{21}V) + I(U_{02}^1U_{02}^2; Y_2|X_2U_{12}U_{21}V) \\ \quad -I(U_{01}^1U_{01}^2; U_{02}^1U_{02}^2|X_1X_2U_{12}U_{21}V) \\ \qquad +I(X_0; Y_0|X_1X_2U_{01}^1U_{02}^1U_{01}^2U_{02}^2U_{12}U_{21}V) \end{array} \right\} \tag{1}$$

where the supremum is over all joint probability mass function $p(x_0, x_1, x_2, u_{01}^1, u_{01}^2, u_{02}^1, u_{02}^2, u_{12}, u_{21}, v)$ on the product set

$$\mathcal{V} \times \mathcal{U}_{12} \times \mathcal{U}_{21} \times \mathcal{U}_{01}^1 \times \mathcal{U}_{01}^2 \times \mathcal{U}_{02}^1 \times \mathcal{U}_{02}^2 \times \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2$$

such that

$$(VU_{12}U_{21}U_{01}^1U_{01}^2U_{02}^1U_{02}^2) \to (X_0X_1X_2) \to Y_0 \tag{2}$$

$$U_{01}^1 \to (X_1, X_0) \to Y_0 \tag{3}$$

$$U_{02}^1 \to (X_2, X_0) \to Y_0 \tag{4}$$

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Proof: For encoding, we make use of regular block Markov superposition encoding and for decoding we make use of backward decoding [16]. The message is divided into three independent parts. The first and second parts are decoded by the first and second relays, respectively, and the receiver can only make estimates of them, whereas the third part is directly decoded by the receiver. The sender and the relays cooperate in the next transmission block to remove the receiver's uncertainly about the first and second parts of the message. The messages at the relays are also divided into two parts, the first part of the message is decoded by the other relay, whereas the second part is directly decoded by the receiver. Fig. 1 shows the individual parts of the messages at the sender and the relays in the proposed coding scheme.

Consider a block Markov encoding scheme with B blocks of transmission, each of n symbols. A sequence of $(B-2)$ messages w_i, $i = 1, 2, \ldots, B-2$, each selected independently and uniformly over $\mathcal{W}$ is to be sent in nB transmissions. (Note that as $B \to \infty$, for fixed n, the rate $R(B-2)/B$ is arbitrarily close to R.) The same codebook is used in each block of transmission. Each relay tries to send the information about the messages to the sink as much as possible through the direct link between the relay and the sink and if there is some more information, through the other relay.

We show that, for any joint probability mass function $p(x_0, x_1, x_2, u_{01}^1, u_{01}^2, u_{02}^1, u_{02}^2, u_{12}, u_{21}, v)$, there exists at least a sequence of codebooks $\mathcal{C}_n$, such that $P_e^{(n)} \to 0$ as $n \to \infty$ if $R \le C$, where

$$R = R_{00} + R_{01}^1 + R_{01}^2 + R_{02}^1 + R_{02}^2 \tag{5}$$

Random coding:

For any joint probability mass function

$$\begin{aligned} &p(x_0, x_1, x_2, u_{01}^1, u_{01}^2, u_{02}^1, u_{02}^2, u_{12}, u_{21}, v) \\ &\quad = p(x_0|x_1, x_2, u_{01}^1, u_{01}^2, u_{02}^1, u_{02}^2, u_{12}, u_{21}, v) \\ &\qquad \times p(u_{01}^1|u_{01}^2, x_1, u_{12}, u_{21}, v)p(u_{02}^1|u_{02}^2, x_2, u_{12}, u_{21}, v) \\ &\qquad \times p(u_{01}^2|u_{12}, u_{21}, v)p(u_{02}^2|u_{12}, u_{21}, v) \\ &\qquad \times p(x_1|u_{12}, v)p(x_2|u_{21}, v)p(u_{12}|v)p(u_{21}|v)p(v) \end{aligned} \tag{6}$$

on the product set

$$\mathcal{V} \times \mathcal{U}_{12} \times \mathcal{U}_{21} \times \mathcal{U}_{01}^1 \times \mathcal{U}_{01}^2 \times \mathcal{U}_{02}^1 \times \mathcal{U}_{02}^2 \times \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2$$

1. Generate $2^{n(R_{01}^2+R_{02}^2)}$ i.i.d. v^n sequences each with probability $p(v^n) = \prod_{i=1}^n p(v_i)$. Index them as $v^n(m_v)$, where $m_v \in [1, 2^{n(R_{01}^2+R_{02}^2)}]$.

2. For each $v^n(m_v)$, generate $2^{nR_{01}^2}$ i.i.d. u_{12}^n sequences each with probability $p(u_{12}^n|v^n) = \prod_{i=1}^n p(u_{12,i}|v_i)$. Index them as $u_{12}^n(m_{12}|m_v)$, where $m_{12} \in [1, 2^{nR_{01}^2}]$.

3. For each $v^n(m_v)$, generate $2^{nR_{02}^2}$ i.i.d. u_{21}^n sequences each with probability $p(u_{21}^n|v^n) = \prod_{i=1}^n p(u_{21,i}|v_i)$. Index them as $u_{21}^n(m_{21}|m_v)$, where $m_{21} \in [1, 2^{nR_{02}^2}]$.

4. For each $v^n(m_v)$, and $u_{12}^n(m_{12}|m_v)$, generate $2^{nR_{01}^1}$ i.i.d. x_1^n sequences each with probability $p(x_1^n|u_{12}^n, v^n) = \prod_{i=1}^n p(x_{1,i}|u_{12,i}, v_i)$. Index them as $x_1^n(m_{10}|m_{12}, m_v)$, where $m_{10} \in [1, 2^{nR_{01}^1}]$.

5. For each $v^n(m_v)$ and $u_{21}^n(m_{21}|m_v)$, generate $2^{nR_{02}^1}$ i.i.d. x_2^n sequences each with probability $p(x_2^n|u_{21}^n, v^n) = \prod_{i=1}^n p(x_{2,i}|u_{21,i}, v_i)$. Index them as $x_2^n(m_{20}|m_{21}, m_v)$, where $m_{20} \in [1, 2^{nR_{02}^1}]$.

6. For each $v^n(m_v)$, $u_{12}^n(m_{12}|m_v)$ and $u_{21}^n(m_{21}|m_v)$, generate $2^{nR_{01}^2}$ i.i.d u_{01}^{2n} each with probability $p(u_{01}^{2n}|u_{12}^n, u_{21}^n, v^n) = \prod_{i=1}^n p(u_{01,i}^2|u_{12,i}, u_{21,i}, v_i)$. Index them as $u_{01}^{2n}(m_{01}^2|m_{12}, m_{21}, m_v)$, where $m_{01}^2 \in [1, 2^{nR_{01}^2}]$.

7. For each $v^n(m_v)$, $u_{12}^n(m_{12}|m_v)$ and $u_{21}^n(m_{21}|m_v)$, generate $2^{nR_{02}^2}$ i.i.d u_{02}^{2n} each with probability $p(u_{02}^{2n}|u_{12}^n, u_{21}^n, v^n) = \prod_{i=1}^n p(u_{02,i}^2|u_{12,i}, u_{21,i}, v_i)$. Index them as $u_{02}^{2n}(m_{02}^2|m_{12}, m_{21}, m_v)$, where $m_{02}^2 \in [1, 2^{nR_{02}^2}]$.

8. For each $v^n(m_v)$, $u_{12}^n(m_{12}|m_v)$, $u_{21}^n(m_{21}|m_v)$, $x_1^n(m_{10}|m_{12}, m_v)$ and $u_{01}^{2n}(m_{01}^2|m_{12}, m_{21}, m_v)$, generate $2^{nR_{01}^1}$ i.i.d u_{01}^{2n} sequences each with probability $p(u_{01}^{1n}|u_{01}^{2n}, x_1^n, u_{12}^n, u_{21}^n, v^n) = \prod_{i=1}^n p(u_{01,i}^1|u_{01,i}^2, x_{1,i}, u_{12,i}, u_{21,i}, v_i)$. Index them as $u_{01}^{1n}(m_{01}^1|m_{01}^2, m_{10}, m_{21}, m_{12}, m_v)$, where $m_{01}^1 \in [1, 2^{nR_{01}^1}]$.

9. For each $v^n(m_v)$, $u_{12}^n(m_{12}|m_v)$, $u_{21}^n(m_{21}|m_v)$, $x_2^n(m_{20}|m_{21}, m_v)$ and $u_{02}^{2n}(m_{02}^2|m_{12}, m_{21}, m_v)$, generate $2^{nR_{02}^1}$ i.i.d u_{02}^{1n} sequences each with probability $p(u_{02}^{1n}|u_{02}^2, x_2^n, u_{12}^n, u_{21}^n, v^n) = \prod_{i=1}^n p(u_{02,i}^1|u_{02,i}^2, x_{2,i}, u_{12,i}, u_{21,i}, v_i)$. Index them as $u_{02}^{1n}(m_{02}^1|m_{02}^2, m_{20}, m_{21}, m_{12}, m_v)$, where $m_{02}^1 \in [1, 2^{nR_{02}^1}]$.

10. For each $v^n(m_v)$, $u_{12}^n(m_{12}|m_v)$, $u_{21}^n(m_{21}|m_v)$, $x_1^n(m_{10}|m_{12}, m_v)$, $x_2^n(m_{20}|m_{21}, m_v)$, $u_{01}^{2n}(m_{01}^2|m_{12}, m_{21}, m_v)$, $u_{02}^{2n}(m_{02}^2|m_{12}, m_{21}, m_v)$, $u_{01}^{1n}(m_{01}^1|m_{01}^2, m_{10}, m_{21}, m_{12}, m_v)$, $u_{02}^{1n}(m_{02}^1|m_{02}^2, m_{20}, m_{21}, m_{12}, m_v)$, generate $2^{nR_{00}}$ i.i.d x_0^n sequences each with probability

$$\begin{aligned} &p(x_0^n|u_{01}^{1n}, u_{02}^{1n}, u_{01}^{2n}, u_{02}^{2n}, x_1^n, x_2^n, u_{12}^n, u_{21}^n, v^n) \\ &\quad = \prod_{i=1}^n p(x_{0,i}|u_{01,i}^1, u_{02,i}^1, u_{01,i}^2, u_{02,i}^2, x_{1,i}, x_{2,i}, u_{12,i}, u_{21,i}, v_i) \end{aligned}$$

Index them as $x_0^n(m_{00,i}|m_{01,i}^1, m_{01,i}^2, m_{02,i}^1, m_{02,i}^2, m_{10,i}, m_{20,i}, m_{12,i}, m_{21,i}, m_{v,i})$, $m_{00} \in [1, 2^{nR_{00}}]$.

The index m_{10} represents the index m_{01}^1 of the previous block. The index m_{20} represents the index m_{02}^1 of the previous block. The index m_{12} represents the index m_{01}^2 of the previous block. The index m_{21} represents the index m_{02}^2 of the previous block. m_v represents the pair indices (m_{12}, m_{21}) of the previous block.

Repeating the above processes (a)–(j) independently, we generate another random codebooks $\mathcal{C}_1$ similar to $\mathcal{C}_0$. We will use these two codebooks in a sequential way as follows: in block $b = 1, \ldots, B$, the codebook $\mathcal{C}_{b \bmod 2}$ is used. Hence, in any two consecutive blocks, codewords from different blocks are independent. This is a property we will use in the analysis of the probability of error.

The codebook and bin assignment are revealed to all parties.

Encoding: Encoding is performed in the following Markov fashion: let $w_i = (m_{00,i}, m_{01,i}^1, m_{01,i}^2, m_{02,i}^1, m_{02,i}^2)$ be the new message to be transmitted in block i. Assume that the first and second relays have estimates of $(\hat{\hat{m}}_{01,i-1}^1, \hat{\hat{m}}_{01,i-1}^2, \hat{\hat{m}}_{21,i-1})$ and $(\hat{\hat{m}}_{02,i-1}^1, \hat{\hat{m}}_{02,i-1}^2, \hat{\hat{m}}_{12,i-1})$ of the previous indices $(m_{01,i-1}^1, m_{01,i-1}^2, m_{21,i-1})$ and $(m_{02,i-1}^1, m_{02,i-1}^2, m_{12,i-1})$, respectively. Since both relays know $(m_{12,i-1}, m_{21,i-1})$, they try to send it to the sink by the cooperation variable v_i. In this fashion, the first and second relays transmissions in block i are $x_1^n(m_{10,i}|m_{12,i}, m_{v,i})$ and $x_2^n(m_{20,i}|m_{21,i}, m_{v,i})$, respectively, where $m_{12,i} = \hat{\hat{m}}_{01,i-1}^2$, $m_{21,i} = \hat{\hat{m}}_{02,i-1}^2$, $m_{10,i} = \hat{\hat{m}}_{01,i-1}^1$ and $m_{20,i} = \hat{\hat{m}}_{02,i-1}^1$. The sender at block i knowing $(m_{01,i}^1, m_{01,i}^2), (m_{02,i}^1, m_{02,i}^2)$ (the parts of the messages transmitted to the relays in block i), $(m_{01,i-1}^1, m_{01,i-1}^2), (m_{02,i-1}^1, m_{02,i-1}^2)$ and hence $m_{10,i}, m_{20,i}, m_{12,i}, m_{21,i}$, transmits $x_0^n(m_{00,i}|m_{01,i}^1, m_{01,i}^2, m_{02,i}^1, m_{02,i}^2, m_{10,i}, m_{20,i}, m_{12,i}, m_{21,i}, m_{v,i})$ implicity including $u_{01}^{1n}(m_{01}^1|m_{01}^2, m_{10}, m_{21}, m_{12}, m_v)$, $u_{02}^{1n}(m_{02}^1|m_{02}^2, m_{20}, m_{21}, m_{12}, m_v)$, $u_{01}^{2n}(m_{01}^2|m_{12}, m_{21}, m_v)$ and $u_{02}^{2n}(m_{02}^2|m_{12}, m_{21}, m_v)$.

Decoding: Assume that at the end of block $(i-1)$, the first relay knows $(m_{01,1}^1, m_{01,2}^1, \ldots, m_{01,(i-1)}^1), (m_{01,1}^2, m_{01,2}^2, \ldots, m_{01,i-1}^2), (m_{10,1}, m_{10,2}, \ldots, m_{10,i}), (m_{12,1}, m_{12,2}, \ldots, m_{12,i}), (m_{21,1}, m_{21,2}, \ldots, m_{21,i-1})$ and $(m_{v,1}, m_{v,2}, \ldots, m_{v,i})$. The second relay knows $(m_{02,1}^1, m_{02,2}^1, \ldots, m_{02,(i-1)}^1)$, $(m_{02,1}^2, m_{02,2}^2, \ldots, m_{02,i-1}^2)$, $(m_{20,1}, m_{20,2}, \ldots, m_{20,i}), (m_{21,1}, m_{21,2}, \ldots, m_{21,i}), (m_{12,1}, m_{12,2}, \ldots, m_{12,i-1})$ and $(m_{v,1}, m_{v,2}, \ldots, m_{v,i})$.

At the end of block i

1. (At the first relay) The first relay declares $\hat{\hat{m}}_{21,i} = m_{21,i}$ or equivalently $\hat{\hat{m}}_{02,i-1}^2 = m_{02,i-1}^2$ was sent by looking for unique index m_{21} such that

$$\begin{pmatrix} u_{21}^n(\hat{\hat{m}}_{21,i}|\hat{\hat{m}}_{v_i}), u_{12}^n(\hat{\hat{m}}_{12,i}|\hat{\hat{m}}_{v_i}), \\ x_1^n(\hat{\hat{m}}_{10,i}|\hat{\hat{m}}_{v_i}, \hat{\hat{m}}_{12,i}), v^n(\hat{\hat{m}}_{v_i}), y_1^n(i) \end{pmatrix} \in A_\epsilon^n$$

and

$$\begin{pmatrix} u_{02}^{2n}(\hat{\hat{m}}_{02,i}^2|\hat{\hat{m}}_{12,i}, \hat{\hat{m}}_{21,i}, \hat{\hat{m}}_{v,i}), \\ u_{21,i}^n(\hat{\hat{m}}_{21,i}|\hat{\hat{m}}_{v_i}), u_{12}^n(\hat{\hat{m}}_{12,i}|\hat{\hat{m}}_{v_i}), \\ x_1^n(\hat{\hat{m}}_{10,i}|\hat{\hat{m}}_{v_i}, \hat{\hat{m}}_{12,i}), v^n(\hat{\hat{m}}_{v_i}), y_1^n(i) \end{pmatrix} \in A_\epsilon^n$$

this can be done with arbitrary small probability of error if n is sufficiently large and

$$R_{02}^2 < I(U_{02}^2 U_{21}; Y_1|X_1 U_{12} V) \qquad (7)$$

The proof can be done by regular encoding/sliding window decoding technique and [11, lemma 1], based on the fact that according to (6), $I(U_{02}^2 U_{21}; X_1 U_{12}|V) = 0$.

2. (At the first relay) The first relay declares $\hat{\hat{m}}_{01,i}^1 = m_{01,i}^1$ and $\hat{\hat{m}}_{01,i}^2 = m_{01,i}^2$ were sent by looking for unique indices m_{01}^1 and m_{01}^2 such that

$$\begin{pmatrix} u_{01}^{1n}(\hat{\hat{m}}_{01,i}^1|\hat{\hat{m}}_{01,i}^2, \hat{\hat{m}}_{10,i}, \hat{\hat{m}}_{12,i}, \hat{\hat{m}}_{21,i}, \hat{\hat{m}}_{v,i}), \\ u_{01}^{2n}(\hat{\hat{m}}_{01,i}^2|\hat{\hat{m}}_{12,i}, \hat{\hat{m}}_{21,i}, \hat{\hat{m}}_{v,i}), \\ x_1^n(\hat{\hat{m}}_{10,i}|\hat{\hat{m}}_{12,i}, \hat{\hat{m}}_{v_i}), u_{21}^n(\hat{\hat{m}}_{21,i}|\hat{\hat{m}}_{v_i}), \\ u_{12}^n(\hat{\hat{m}}_{12,i}|\hat{\hat{m}}_{v_i}), v^n(\hat{\hat{m}}_{v_i}), y_1^n(i) \end{pmatrix} \in A_\epsilon^n$$

this can be done with arbitrary small probability of error if n is sufficiently large and

$$R_{01}^1 + R_{01}^2 < I(U_{01}^1 U_{01}^2; Y_1|X_1 U_{12} U_{21} V) \qquad (8)$$

The proof is based on joint decoding of $(\hat{\hat{m}}_{01}^1, \hat{\hat{m}}_{01}^2)$, in this way the probability of error is expressed as

$$\begin{aligned} P_e^n = & \sum_{(m_{01}^1, m_{01}^2) \in [1, 2^{nR_{01}^1}] \times [1, 2^{nR_{01}^2}]} \ \sum_{(u_{01}^1, u_{01}^2, x_1, u_{12}, u_{21}, v, y_1) \in A_\epsilon^n} \\ & \times \begin{pmatrix} p(u_{01}^1|u_{01}^2, x_1, u_{12}, u_{21}, v) p(u_{01}^2|u_{12}, u_{21}, v) \\ p(y_1|x_1, u_{12}, u_{21}, v) p(x_1, u_{12}, u_{21}, v) \end{pmatrix} \\ = & 2^{n(R_{01}^1 + R_{01}^2)} \|A_\epsilon^n\| \ 2^{-n(H(U_{01}^1|U_{01}^2 X_1 U_{12} U_{21} V) - \epsilon)} \\ & \times 2^{-n(H(U_{01}^2|U_{12} U_{21} V) - \epsilon)} 2^{-n(H(Y_1|X_1 U_{12} U_{21} V) - \epsilon)} \\ & \times 2^{-n(H(X_1 U_{12} U_{21} V) - \epsilon)} \\ = & 2^{n(R_{01}^1 + R_{01}^2)} 2^{n(H(Y_1 U_{01}^1 U_{01}^2 X_1 U_{12} U_{21} V) + \epsilon)} \\ & \times 2^{-n(H(U_{01}^1|U_{01}^2 X_1 U_{12} U_{21} V) - \epsilon)} 2^{-n(H(U_{01}^2|U_{12} U_{21} V) - \epsilon)} \\ & \times 2^{-n(H(Y_1|X_1 U_{12} U_{21} V) - \epsilon)} 2^{-n(H(X_1 U_{12} U_{21} V) - \epsilon)} \\ \overset{(a)}{=} & 2^{n(R_{01}^1 + R_{01}^2)} 2^{n(H(Y_1 U_{01}^1 U_{01}^2 X_1 U_{12} U_{21} V) + \epsilon)} \\ & \times 2^{-n(H(U_{01}^1|U_{01}^2 X_1 U_{12} U_{21} V) - \epsilon)} 2^{-n(H(U_{01}^2|X_1 U_{12} U_{21} V) - \epsilon)} \\ & \times 2^{-n(H(Y_1|X_1 U_{12} U_{21} V) - \epsilon)} 2^{-n(H(X_1 U_{12} U_{21} V) - \epsilon)} \\ = & 2^{n(R_{01}^1 + R_{01}^2)} 2^{-n(I(U_{01}^1 U_{01}^2; Y_1|X_1 U_{12} U_{21} V) - 5\epsilon)} \end{aligned}$$

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where [1] follows from the fact that according to (6), $I(U_{01}^2; X_1|U_{12}U_{21}V) = 0$. Since $\epsilon > 0$ is arbitrary, (8) imply that $P_e \to 0$ as $n \to \infty$.

3. (At the second relay) Because of the symmetry in the definition of codewords by the same argument as the previous step, the following rate region is obtained

$$R_{01}^2 < I(U_{01}^2 U_{12}; Y_2|X_2 U_{21} V) \quad (9)$$

$$R_{02}^1 + R_{02}^2 < I(U_{02}^1 U_{02}^2; Y_2|X_2 U_{12} U_{21} V) \quad (10)$$

4. (Jointly typical $(u_{01}^{1n}, u_{01}^{2n}), (u_{02}^{1n}, u_{02}^{2n})$) For each product bin $[1, 2^{n(R_{01}^1+R_{01}^2)}] \times [1, 2^{n(R_{02}^1+R_{02}^2)}]$, jointly typical pair $((u_{01}^{1n}, u_{01}^{2n}), (u_{02}^{1n}, u_{02}^{2n}))$ can be found if

$$\begin{aligned} &R_{01}^1 + R_{01}^2 + R_{02}^1 + R_{02}^2 \\ &< (I(U_{01}^1 U_{01}^2; Y_1|X_1 U_{12} U_{21} V) + I(U_{02}^1 U_{02}^2; Y_2|X_2 U_{12} U_{21} V) \\ &-I(U_{01}^1 U_{01}^2; U_{02}^1 U_{02}^2|X_1 X_2 U_{12} U_{21} V) \end{aligned} \quad (11)$$

The proof can be done based on product binning method as mentioned for the proof of Marton's rate region for broadcast channel in [15]. Recall that, given $u_{12}^n, u_{21}^n, x_1^n$ and x_2^n, independent choices of $(u_{01}^{1n}, u_{01}^{2n})$ and $(u_{02}^{1n}, u_{02}^{2n})$ result in a jointly typical $((u_{01}^{1n}, u_{01}^{2n}), (u_{02}^{1n}, u_{02}^{2n}))$ with probability $2^{-nI(U_{01}^1 U_{01}^2; U_{02}^1 U_{02}^2|X_1 X_2 U_{12} U_{21} V)}$. Now according to (8) and (10), there are $2^{n(I(U_{01}^1 U_{01}^2; Y_1|X_1 U_{12} U_{21} V) - R_{01}^1 - R_{01}^2)}$, $(u_{01}^{1n}, u_{01}^{2n})$'s in any bin $U_{01}^1 \times U_{01}^2$ and $2^{n(I(U_{02}^1 U_{02}^2; Y_2|X_2 U_{12} U_{21} V) - R_{02}^1 - R_{02}^2)}$, $(u_{02}^{1n}, u_{02}^{2n})$'s in any bin $U_{02}^1 \times U_{02}^2$. Thus, the expected number of jointly typical $(u_{01}^{1n}, u_{01}^{2n}, u_{02}^{1n}, u_{02}^{2n})$ in a given product bin $\mathcal{U}_{01}^1 \times \mathcal{U}_{02}^1 \times \mathcal{U}_{02}^1 \times \mathcal{U}_{02}^2$ is

$$\begin{aligned} &2^{n(I(U_{01}^1 U_{01}^2; Y_1|X_1 U_{12} U_{21} V) - R_{01}^1 - R_{01}^2)} \\ &\times 2^{n(I(U_{02}^1 U_{02}^2; Y_2|X_2 U_{12} U_{21} V) - R_{02}^1 - R_{02}^2)} \\ &\times 2^{-nI(U_{01}^1 U_{01}^2; U_{02}^1 U_{02}^2|X_1 X_2 U_{12} U_{21} V)} \end{aligned}$$

The desired jointly typical $(u_{01}^{1n}, u_{01}^{2n}, u_{02}^{1n}, u_{02}^{2n})$ can be found if this expected number is much greater than 1, which follows if satisfies (11).

5. (Backward decoding at the destination) Assume that $\hat{m}_{01,i}^2, \hat{m}_{02,i}^2, \hat{m}_{01,i-1}^2$ and $\hat{m}_{02,i-1}^2$ have been decoded accurately. The sink determines $\hat{m}_{01,i-2}^2$ and $\hat{m}_{02,i-2}^2$ were sent such that

$$\begin{pmatrix} u_{01}^{2n}(\hat{m}_{01,i}^2|\hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ u_{02}^{2n}(\hat{m}_{02,i}^2|\hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ u_{12}^n(\hat{m}_{12,i}|\hat{m}_{v,i}), \\ u_{21}^n(\hat{m}_{21,i}|\hat{m}_{v,i}), v^n(\hat{m}_{v_i}), y_0^n(i) \end{pmatrix} \in A_\epsilon^n \quad (12)$$

where $\hat{m}_{12,i} = \hat{m}_{01,i-1}^2$, $\hat{m}_{21,i} = \hat{m}_{02,i-1}^2$ and $\hat{m}_{v,i} \triangleq (\hat{m}_{01,i-2}^2, \hat{m}_{02,i-2}^2)$. This can be done with arbitrary small probability of error if n is sufficiently large and

$$R_{01}^2 + R_{02}^2 < I(U_{01}^2 U_{02}^2 U_{12} U_{21} V; Y_0) \quad (13)$$

6. (Backward decoding at the destination) By knowing $\hat{m}_{v,i}, \hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{01,i}^2$ and $\hat{m}_{02,i}^2$ and assuming that $\hat{m}_{01,i}^1$ and $\hat{m}_{02,i}^1$ have been decoded accurately, the sink determines $\hat{m}_{01,i-1}^1 = m_{01,i-1}^1$ and $\hat{m}_{02,i-1}^1 = m_{02,i-1}^1$ were sent such that

$$\begin{pmatrix} u_{01}^{1n}(\hat{m}_{01,i}^1|\hat{m}_{01,i}^2, \hat{m}_{10,i}, \hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ u_{02}^{1n}(\hat{m}_{02,i}^1|\hat{m}_{02,i}^2, \hat{m}_{20,i}, \hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ x_1^n(\hat{m}_{10,i}|\hat{m}_{v_i}, \hat{m}_{12,i}), x_2^n(\hat{m}_{20,i}|\hat{m}_{v_i}, \hat{m}_{21,i}), \\ u_{01}^{2n}(\hat{m}_{01,i}^2|\hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ u_{02}^{2n}(\hat{m}_{02,i}^2|\hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ u_{12}^n(\hat{m}_{12,i}|\hat{m}_{v,i}), u_{21}^n(\hat{m}_{21,i}|\hat{m}_{v,i}), v^n(\hat{m}_{v_i}), y_0^n(i) \end{pmatrix} \in A_\epsilon^n$$

where $\hat{m}_{10,i} = \hat{m}_{01,i-1}^1$ and $\hat{m}_{20,i} = \hat{m}_{02,i-1}^1$. This can be done with arbitrary small probability of error if n is sufficiently large and

$$R_{01}^1 < I(X_1 U_{01}^1; Y_0|X_2 U_{02}^1, U_{01}^2 U_{02}^2 U_{12} U_{21} V) \quad (14)$$

$$R_{02}^1 < I(X_2 U_{02}^1; Y_0|X_1 U_{01}^1, U_{01}^2 U_{02}^2 U_{12} U_{21} V) \quad (15)$$

$$R_{01}^1 + R_{02}^1 < I(X_1 X_2 U_{01}^1 U_{02}^1; Y_0|U_{01}^2 U_{02}^2 U_{12} U_{21} V) \quad (16)$$

The proof is the same as the proof of the capacity of the multiple access channel [16], based on the fact that according to (6), (X_1, U_{01}^1) and (X_2, U_{02}^1) are independent given $(U_{01}^2 U_{02}^2 U_{12} U_{21} V)$.

7. (At the destination) Knowing By knowing $\hat{m}_{v,i}, \hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{01,i}^2, \hat{m}_{02,i}^2, \hat{m}_{01,i}^1, \hat{m}_{02,i}^1$, the sink declares $\hat{m}_{00,i} = m_{00,i}$ is sent if it is a unique index $m_{00,i}$ such that

$$\begin{pmatrix} x_0^n(\hat{m}_{00,i}|\hat{m}_{01,i}^1, \hat{m}_{01,i}^2, \hat{m}_{02,i}^1, \hat{m}_{02,i}^2, \hat{m}_{10,i}, \hat{m}_{20,i}, \\ \hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ u_{01}^{1n}(\hat{m}_{01,i}^1|\hat{m}_{01,i}^2, \hat{m}_{10,i}, \hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ u_{02}^{1n}(\hat{m}_{02,i}^1|\hat{m}_{02,i}^2, \hat{m}_{20,i}, \hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ x_1^n(\hat{m}_{10,i}|\hat{m}_{v_i}, \hat{m}_{12,i}), x_2^n(\hat{m}_{20,i}|\hat{m}_{v_i}, \hat{m}_{21,i}), \\ u_{01}^{2n}(\hat{m}_{01,i}^2|\hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ u_{02}^{2n}(\hat{m}_{02,i}^2|\hat{m}_{12,i}, \hat{m}_{21,i}, \hat{m}_{v,i}), \\ u_{12}^n(\hat{m}_{12,i}|\hat{m}_{v,i}), u_{21}^n(\hat{m}_{21,i}|\hat{m}_{v,i}), v^n(\hat{m}_{v_i}), y_0^n(i) \end{pmatrix} \in A_\epsilon^n$$

This can be done with arbitrary small probability of error if n is sufficiently large and

$$R_{00} < I(X_0; Y_0|X_1 X_2 U_{01}^1 U_{02}^1 U_{01}^2 U_{02}^2 U_{12} U_{21} V) \quad (17)$$

From the above constraints, we obtain the following results.

Equations (2), (5), (13), (16) and (17) results the first term of (1).

Equations (3), (5), (7), (8), (15) and (17) results the second term of (1).

Equations (4), (5), (9), (10), (14) and (17) results the third term of (1).

Equations (4), (9), (7), (11) and (17) results the forth term of (1).

Equations (5), (11) and (17) results the fifth term of (1).

This completes the proof of Theorem 1. □

Remark: 1. By substituting $U_{01}^1 = U_{01}^2 = X_0, U_{12} = X_1, V = X_2$ and $U_{02}^1 = U_{02}^2 = U_{21} = 0$ in (1), the proposed rate in [7], the best proposed rate until now based on decode-and-forward method, is obtained as follows

$$C \geq \sup_{p(x_0,x_1,x_2)} \min\{I(X_0X_1X_2; Y_0), I(X_0X_1; Y_2|X_2), I(X_0; Y_1|X_1X_2)\} \tag{18}$$

3 A class of semi-deterministic relay network

In this section, we first give a review of Aref network, then we introduce a generalisation of Aref network named semi-deterministic relay networks with no interference at the relays. We show that the capacity of such relay networks is obtained using the proposed achievable rate of previous section (1), that is, it coincides the max flow min cut upper bound.

3.1 Deterministic relay network with no interference

A discrete memoryless relay network is called deterministic if the output of each branch is a deterministic function of its input. Specifically, there exist functions $h_j(.)$ such that

$$h_j: \mathcal{X}_0 \times \cdots \times \mathcal{X}_N \rightarrow \mathcal{Y}_j, \quad 0 \leq j \leq N$$

The sink symbol $y_0 = h_0(x_0, \ldots, x_N)$, and the relay symbols $y_j = h_j(x_0, \ldots, x_N)$, $1 \leq j \leq N$, is deterministically related to the senders symbols $(x_0, \ldots, x_N)$.

A deterministic relay network with no interference, e.g. Aref network, is depicted in Fig. 2, in which $y_{ij} = h_{ij}(x_i)$, $i \neq j$ (except for $i = j = 0$) is the received signal at the jth node corresponding to the ith node sender. In general, each node receives N signals [the sink receives $(N+1)$ signals], i.e.

$$y_j \triangleq (y_{0j}, \ldots, y_{kj}, \ldots, y_{Nj}), \quad 0 \leq j \leq N, \quad k \neq j, \quad \text{(except for } k = j = 0)$$

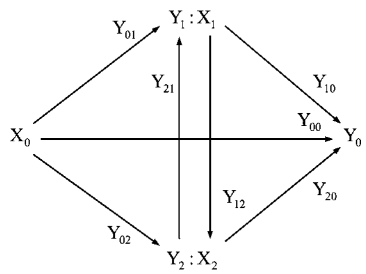

Figure 2 *Deterministic relay networks with no interference [4]*

is a vector with N entries which show the received signals at the jth node.

Aref determined the unicast capacity of the above networks by showing that a protocol that uses binning [17, Sec. 15.4] at the physical layer and routing at the network layer achieves an information-theoretic cut-set bound [17, Sec. 15.10] that generalises the usual cut-set bound of networking theory. Aref proved the following theorem with $N = 2$ and generalised the formulation to the network with N intermediate nodes.

Theorem 2 [4 Sec. 3.5]: For the network depicted in Fig. 2, the capacity is given by

$$C = \sup_{p(x_0)p(x_1)p(x_2)} \min\{H(Y_{00}, Y_{01}, Y_{02}), H(Y_{00}, Y_{01}) + H(Y_{20}, Y_{21}), H(Y_{00}, Y_{02}) + H(Y_{10}, Y_{12}), H(Y_{00}) + H(Y_{10}) + H(Y_{20})\} \tag{19}$$

In Aref networks, the communication between different nodes is done through physically separated channels (wires). This means that the communication between two particular nodes does not affect the communication between others. In this setup, the maximum achievable rate is given by the minimum cut-capacity over all cuts separating the source nodes and a destination node. Because of the special structure of wireline networks, the cut-capacity for any cut is equal to the sum of the capacities of the channels crossing the cut.

3.2 Semi-deterministic relay networks with no interference at the relays

In this section we define an alternation of Aref networks. It has the following differences with Aref networks.

- The channels at the multiple access side of the network are not deterministic.
- Y_{01} and Y_{02} in the defined network are deterministic functions not only of X_0, but also of their outputs, i.e. $Y_{0i} = h_{0i}(x_0, x_i)$ for $i = 1, 2$.

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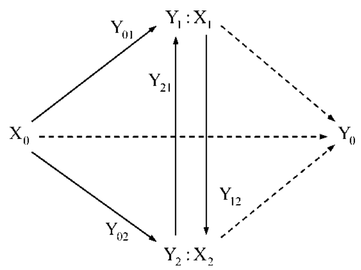

Figure 3 *Semi deterministic relay networks with no interference at the relays*

- The assumption of no interferencing is relaxed at the sink.
- Only the relay inputs are assumed independent.

Consider the network of Fig. 3, in which

$$Y_{12} = h_{12}(x_1),\ Y_{21} = h_{21}(x_2),\ Y_{01} = h_{01}(x_0, x_1),\ Y_{02} = h_{02}(x_0, x_2) \quad (20)$$

where h_{12}, h_{21}, h_{01} and h_{02} are deterministic functions. In this figure, random links are shown with dash lines.

We name this network as semi-deterministic relay network with no interference at the relays. Next, we prove the capacity of this network.

Theorem 3: The capacity of semi-deterministic relay network shown in Fig. 3 is given by

$$\begin{aligned} C = \sup_{p(x_0|x_1,x_2)\,p(x_1)\,p(x_2)} \min\{ & I(X_0X_1X_2; Y_0), \\ & H(Y_{01}Y_{21}|X_1) + I(X_0X_2; Y_0|X_1Y_{01}Y_{21}), \\ & H(Y_{02}Y_{12}|X_2) + I(X_0X_1; Y_0|X_2Y_{02}Y_{12}), \\ & H(Y_{01}Y_{02}|X_1X_2) + I(X_0; Y_0|Y_{01}Y_{02}X_1X_2)\} \end{aligned} \quad (21)$$

Proof: The proof of achievability part of this theorem follows by replacing $U_{01}^1 = Y_{01}$, $U_{02}^1 = Y_{02}$, $U_{01}^2 = U_{02}^2 = 0$, $U_{12} = Y_{12}$ and $U_{21} = Y_{21}$ and $V = 0$ in (1) and by respect to (20), (6) reduces to

$$\begin{aligned} p(x_0, x_1, x_2, y_{01}, y_{02}, y_{12}, y_{21}) = {} & p(x_0|y_{01}, y_{02}, x_1, x_2, y_{12}, y_{21}) \\ & p(y_{01}|x_1, y_{12}, y_{21})p(y_{02}|x_2, y_{12}, y_{21})p(x_1)p(x_2) \end{aligned} \quad (22)$$

and the following are true.

$$\begin{aligned} & I(U_{01}^1U_{01}^2; Y_1|X_1U_{12}U_{21}V) + I(U_{02}^2U_{21}; Y_1|X_1U_{12}V) \\ & + I(X_0X_2; Y_0|X_1U_{01}^1U_{01}^2U_{02}^2U_{12}U_{21}V) \\ & = I(Y_{01}; Y_{01}Y_{21}|X_1Y_{12}Y_{21}) + I(Y_{21}; Y_{01}Y_{21}|X_1Y_{12}) \\ & \quad + I(X_0X_2; Y_0|X_1Y_{01}Y_{12}Y_{21}) \\ & = H(Y_{01}Y_{21}|X_1Y_{12}) + I(X_0X_2; Y_0|X_1Y_{01}Y_{21}) \end{aligned} \quad (23)$$

$$\begin{aligned} & I(U_{01}^1U_{01}^2; Y_1|X_1U_{12}U_{21}V) + I(U_{02}^1U_{02}^2; Y_2|X_2U_{12}U_{21}V) \\ & - I(U_{01}^1U_{01}^2; U_{02}^1U_{02}^2|X_1X_2U_{12}U_{21}V) \\ & + I(X_0; Y_0|X_1X_2U_{01}^1U_{02}^1U_{01}^2U_{02}^2U_{12}U_{21}V)\} \\ & = I(Y_{01}; Y_{01}Y_{21}|X_1Y_{12}Y_{21}) + I(Y_{02}; Y_{02}Y_{12}|X_2Y_{12}Y_{21}) \\ & - I(Y_{01}; Y_{02}|X_1X_2Y_{12}Y_{21}) \\ & + I(X_0; Y_0|X_1X_2Y_{01}Y_{02}Y_{12}Y_{21}) \\ & = H(Y_{01}|X_1Y_{12}Y_{21}) + H(Y_{02}|X_2Y_{12}Y_{21}) \\ & - I(Y_{01}; Y_{02}|X_1X_2Y_{12}Y_{21}) \\ & + I(X_0; Y_0|X_1X_2Y_{01}Y_{02}Y_{12}Y_{21}) \end{aligned} \quad (24)$$

$$\begin{aligned} & \overset{(a)}{=} H(Y_{01}|X_1Y_{12}Y_{21}X_2) + H(Y_{02}|X_2Y_{12}Y_{21}X_1) \\ & - I(Y_{01}; Y_{02}|X_1X_2Y_{12}Y_{21}) \\ & + I(X_0; Y_0|X_1X_2Y_{01}Y_{02}Y_{12}Y_{21}) \\ & = H(Y_{01}Y_{02}|X_1X_2) + I(X_0; Y_0|X_1X_2Y_{01}Y_{02}) \end{aligned}$$

$$\begin{aligned} & I(U_{02}^2U_{21}; Y_1|X_1U_{12}V) + I(U_{01}^2U_{12}; Y_2|X_2U_{21}V) \\ & + I(X_0X_1X_2; Y_0|U_{01}^2U_{02}^2U_{12}U_{21}V) \\ & = I(Y_{21}; Y_{01}Y_{21}|X_1Y_{12}) + I(Y_{12}; Y_{02}Y_{12}|X_2Y_{21}) \\ & + I(X_0X_1X_2; Y_0|Y_{12}Y_{21}) \\ & = H(Y_{21}|X_1Y_{12}) + H(Y_{12}|X_2Y_{21}) \\ & + I(X_0X_1X_2; Y_0|Y_{12}Y_{21}) \end{aligned} \quad (25)$$

$$\begin{aligned} & \overset{(b)}{=} H(Y_{21}) + H(Y_{12}) + I(X_0X_1X_2; Y_0|Y_{12}Y_{21}) \\ & = H(Y_{21}) + H(Y_{12}) + H(Y_0|Y_{12}Y_{21}) - H(Y_0|X_0X_1X_2) \\ & = H(Y_0Y_{12}Y_{21}) - H(Y_0|X_0X_1X_2) \\ & = H(Y_{12}Y_{21}|Y_0) + I(X_0X_1X_2; Y_0) \end{aligned}$$

where (a) follows from the fact that according to (22), it is concluded that given (X_1, Y_{21}), Y_{01} and X_2 are independent and given (X_2, Y_{12}), Y_{02} and X_1 are independent. (b) follows from the fact that X_1 and X_2 are dependent.

From the above constraints, it is observed that (25) is greater than the first term of (1), thus it is omitted. Equation (23) results the second term of (21). Equation (24) results the fourth term of (21). Because of the symmetry of the network, by the same argument as (23), the third term of (21) is resulted from the third term of (1).

The converse part of this theorem follows from max-flow min-cut theorem. For the network with two relay this bound reduces to

$$\begin{aligned} C \le \sup_{p(x_0,x_1,x_2)} \min\{ & I(X_0X_1X_2; Y_0), I(X_0; Y_0Y_1Y_2|X_1X_2), \\ & I(X_0X_1; Y_0Y_2|X_2), I(X_0X_2; Y_0Y_1|X_1)\} \end{aligned} \quad (26)$$

with respect to (20) and (26), the following are true.

$$\begin{aligned} I(X_0; Y_0Y_1Y_2|X_1X_2) &\overset{(a)}{=} I(X_0; Y_1Y_2|X_1X_2) \\ &\quad + I(X_0; Y_0|X_1X_2Y_1Y_2) \\ &\overset{(b)}{=} H(Y_1Y_2|X_1X_2) \\ &\quad + I(X_0; Y_0|Y_1Y_2X_1X_2) \\ &\overset{(c)}{=} H(Y_{01}Y_{02}|X_1X_2) \\ &\quad + I(X_0; Y_0|Y_{01}Y_{02}X_1X_2) \\ I(X_0X_2; Y_0Y_1|X_1) &\overset{(d)}{=} I(X_0X_2; Y_1|X_1) \\ &\quad + I(X_0X_2; Y_0|X_1Y_1) \\ &\overset{(e)}{=} H(Y_{01}Y_{21}|X_1) \\ &\quad + I(X_0X_2; Y_0|X_1Y_{01}Y_{21}) \end{aligned}$$

Because of the symmetry of the network, by the same reason as previously we have,

$$I(X_0X_1; Y_0Y_2|X_2) = H(Y_{02}Y_{12}|X_2) + I(X_0X_1; Y_0|X_2Y_{02}Y_{12})$$

where (a) and (d) follow from the chain rule for information. (b), (c) and (e) follows from (20).

This completes the proof of Theorem 3. □

The application of Aref network is very well understood in network coding that considers the networks with deterministic links. The application of 'semi-deterministic relay networks with no interference at the relays and independent relay inputs' is in the relay networks that consist of both deterministic and non-deterministic links (wire-line and wireless links). The concept of no interferencing can be applied to the wireless scenarios in which the relays cannot send and receive in the same frequency band.

4 Conclusion

In this paper, we have derived a new achievable rate for two-relay networks based on symmetric relaying and partial decoding, in which, each relay can partially decode the message of the other relay and the source message. This concept can be applied to the relay networks such as Aref network in which no priorities are considered for any of the relays over the others in receiving the message of the source and the message of the other relays. The strategy make use of new techniques such as random binning [15], regular encoding/sliding window decoding [7] and regular encoding/backward decoding [16]. Based on the proposed achievable rate, we obtain the capacity of a generalisation of Aref networks called semi-deterministic relay networks with no interference at the relays and independent relay inputs. This is a timely subject, given the current interest in relay channels and network coding.

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